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RESEARCH

Analysis of a C^0 finite element method for the biharmonic problem with Dirichlet boundary conditions

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Abstract

The main focus of this paper is to approximate the biharmonic equation with Dirichlet boundary conditions in a polygonal domain by decomposing it into a system of second-order equations. Subsequently, we explore the regularities exhibited by these equations in each system. Upon demonstrating that the solutions of each resulting system are equivalent to those of the original fourth-order problem in both convex and non-convex polygonal domains, we introduce C^0 finite element algorithms designed to solve the decoupled system, accompanied by a comprehensive analysis of error estimates. In contrast to the biharmonic problem, the solutions of the Poisson and Stokes problems display lower regularities, leading to diminished convergence rates for their finite element approximations. This can, in turn, impact the overall convergence rate of the finite element approximation on quasi-uniform meshes for the biharmonic problem. However, we establish an invariant relationship for the source term in the Stokes equation, showing that, under appropriate conditions, the convergence rate of the biharmonic approximation is solely influenced by the Stokes approximation, rather than the first Poisson approximation. To recover the optimal convergence rate for the biharmonic approximation, we also explore the regularities in the weighted Sobolev space and introduce the graded finite element method with the grading parameter only governed by the last Poisson equation. To validate our theoretical insights, we present numerical test results.

Keywords Biharmonic equation · Stokes equation · Poisson equation · Taylor-Hood method · Error estimates

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1 Introduction

Consider the biharmonic problem

$$\Delta^2\phi = f \quad \text{in } \Omega, \qquad \phi = \partial_{\mathbf{n}}\phi = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a simply connected polygonal domain, $\mathbf{n}$ is the outward normal derivative, and f is a given function. The boundary conditions in (1.1) are referred to as the homogeneous Dirichlet boundary conditions or clamped boundary conditions [26, 53] that occur, for example, in fluid mechanics [31] and linear elasticity [22]. Equation (1.1) is a fourth-order elliptic equation for which the variational form involves second-order differential operators. For conforming finite element methods, these high-order derivatives call for finite element spaces that belong to H^2, such as the C^1 Argyris finite element method [3]. It is possible to relax the constraint on the global regularity of the finite element space. Such methods include various nonconforming methods (for example, the Morley finite element method [47, 56], the interior penalty method [12], and the DG method [29]). We also mention the mixed finite element methods [23, 25, 46] that utilize the C^0 Lagrange finite elements. These mixed methods, however, require regularity assumptions on the solution that usually lead to restrictive conditions on the domain geometry.

To obtain a mixed formulation, one can first decompose (1.1) into a system of low-order equations, and then apply appropriate numerical methods to solve the system [8, 27, 37, 49]. For example, (1.1) can be decoupled into two Poisson problems by introducing the intermediate function $\psi = -\Delta\phi$. Different from the biharmonic equation with Navier boundary conditions ($\phi = \Delta\phi = 0$ on $\partial\Omega$) that allows one to obtain two Poisson equations that are completely decoupled [45], applying such decomposition to (1.1) leads to two Poisson equations with either underdetermined or overdetermined boundary conditions,

$$\begin{cases} -\Delta\psi = f & \text{in } \Omega, \\ \text{no data} & \text{on } \partial\Omega; \end{cases} \qquad \text{and} \qquad \begin{cases} -\Delta\phi = \psi & \text{in } \Omega, \\ \phi = \partial_{\mathbf{n}}\phi = 0 & \text{on } \partial\Omega. \end{cases}$$

To overcome this difficulty, Ciarlet and Raviart [23] proposed a mixed finite element method for (1.1) in a smooth domain by introducing a conditioned function space. Later, Monk [46] improved this mixed method to allow an $H^1(\Omega)$ intermediate function ψ. The main difference between these two works lies in the smoothness of the intermediate function ψ, while the solutions are equivalent under some smoothness assumptions [25]. However, these two mixed methods can only be applied to smooth or convex polygonal domains. For non-convex polygonal domains, the solution of the Ciarlet-Raviart mixed method is equivalent to the weak solution of (1.1), but the corresponding finite element approximations do not converge to the exact solution due to the low regularity of the intermediate function [23, 25]. The Monk mixed method increased the regularity of the intermediate function, but it can result in spurious solutions in non-convex polygonal domains. An augmented method was introduced in [25] to remove these spurious solutions. In addition, we refer the readers to the related mixed method in [24], and the work in [57] on preconditioning techniques for the discrete system, both of which are based on the Ciarlet-Raviart formulation [23].

In this paper, we study efficient C^0 finite element methods for (1.1) applicable to general polygonal domains, motivated by the intrinsic connection between the Stokes problem and the biharmonic problem. We start by identifying a suitable source term for the Stokes problem, either by solving a Poisson equation with f as the right-hand side or manually selecting one with curl depending on f. While the source term isn't unique, the velocity vector in the Stokes problem uniquely depends on f. We then obtain the solution of the biharmonic equation by solving another Poisson equation, using the curl of the Stokes problem's velocity vector as the right-hand side. We use two decomposition methods: the S-P system (one Stokes problem and one Poisson problem) or the well-known P-S-P system (two Poisson problems and one Stokes problem). For each decoupled system, the solution of the last Poisson equation recasts the solution of the biharmonic problem, and we demonstrate that the solution of each decoupled system is equivalent to the solution of the biharmonic problem (1.1), applicable to both convex and non-convex polygonal domains. We carefully derive the regularity of the solution of each involved equation in the system in a polygonal domain. The P-S-P system was also exploited in [5, 18, 19, 21, 28, 36] for (1.1).

We derive the regularity of the solution for each equation in the P-S-P and S-P systems. For each decoupled system, we provide a C^0 finite element algorithm, in which the MINI element [4] or the Taylor-Hood element [13, 31, 52] is used to solve the Stokes problem, and the Lagrange element [14, 22] is used for the Poisson problems. For each algorithm, we demonstrate the convergence of the proposed finite element solution to the solution of the biharmonic problem and conduct error analysis on quasi-uniform meshes. For the P-S-P decomposition in a polygonal domain, depending on the largest interior angle of the polygonal domain, both the solutions of the first Poisson equation and of the Stokes equation may exhibit low regularities. Consequently, the finite element approximations can yield lower convergence rates, adversely affecting the overall convergence rate of the finite element approximation for the biharmonic problem. Based on the standard error estimate for the Poisson and the Stokes problems, the convergence rate of the biharmonic approximation is influenced by both the convergence rates of the Stokes approximation and the first Poisson approximation. However, thanks to an invariant relationship established in Lemma 3.6 concerning the source term of the Stokes equation, we obtain an improved $H^1(\Omega)$ convergence rate for the biharmonic approximation.

In our case of the P-S-P decomposition, as long as the polynomial degree does not exceed 4, the convergence rate of the Stokes approximation can not be influenced by that of the first Poisson approximation. In particular, the $H^1(\Omega)$ convergence rate of the biharmonic approximation shows the same convergence rate as the $[L^2(\Omega)]^2$ error of velocity approximation in the Stokes equation. In this work, we add new techniques in the error analysis and obtain an improved H^1 convergence rate compared with the standard error analysis, see Corollary 3.11 for a comparison with the existing result using standard error analysis. Our findings reveal that the biharmonic approximation obtained in this work exhibits a higher-order H^1 convergence rate and can be numerically corroborated. Additionally, when the polynomial degree does not exceed 4, the convergence rate of the biharmonic approximation based on the P-S-P decomposition is identical to that obtained through the S-P decomposition. However, if the polynomial degree exceeds 4, the latter one yields a higher convergence rate due to its independence from an extra Poisson equation.

We also establish regularity estimates for each decoupled system within a class of Kondratiev-type weighted Sobolev spaces based on the general regularity theory in [33, 34, 48]. Leveraging these regularity results, we propose graded mesh refinement algorithms. However, the different equations, two Poisson equations and one Stokes equation, in the system all show different regularities, which increases the difficulty in choosing the grading parameter for the meshes. In our graded finite element algorithms, the grading parameter is only governed by the last Possion equation, not the Stokes equation, and the resulting algorithms enable the associated biharmonic approximation to achieve optimal convergence rates. It's worth noting that while the biharmonic approximation attains the optimal convergence rate, the Stokes approximation doesn't necessarily need to be optimal. Similar to the error estimates on quasi-uniform meshes, the convergence rates of biharmonic approximations for both the P-S-P system and the S-P system are identical when the polynomial degree does not exceed 4. As a by-product of this research, we additionally identify the graded meshes that lead to optimal convergence rates for finite element approximations of the Stokes problem, utilizing either the MINI element or the Taylor-Hood element.

The rest of the paper is organized as follows. In Section 2, based on the general regularity theory for second-order elliptic equations and the Stokes equation, we review the weak solutions of all involved equations and show the equivalence between the solution of the proposed system and that of the biharmonic problem. In Section 3, we propose two finite element algorithms and obtain error estimates on quasi-uniform meshes. In Section 4, we introduce the weighted Sobolev space and derive regularity estimates for the solution. We also present the graded mesh algorithm and provide optimal error estimates on graded meshes. We report numerical test results in Section 5 to validate the theory.

In this paper, the generic constant $C > 0$ in our estimates may vary but depends solely on the computational domain, remaining independent of the functions involved and the mesh level in the finite element algorithms.

2 The biharmonic problem and its decoupled formulation

2.1 Well-posedness and regularity of the solution

Let $\gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbb{Z}^d_{\geq 0}$ be a multi-index such that $\partial^\gamma := \partial_1^{\gamma_1} \cdots \partial_{x_d}^{\gamma_d}$ and $|\gamma| := \sum_{i=1}^d \gamma_i$. Defined by $H^m(\Omega)$, $m \geq 0$, the Sobolev space that consists of functions whose derivatives corresponding to the multi-index γ are square integrable. Denote by $H_0^1(\Omega) \subset H^1(\Omega)$ the subspace consisting of functions with zero trace on the boundary $\partial\Omega$. Let $L^2(\Omega) := H^0(\Omega)$. We denote the norm $\|\cdot\|_{L^2(\Omega)}$ by $\|\cdot\|$ when there is no ambiguity about the underlying domain. For $s > 0$, let $s = m + t$, where

$m \in \mathbb{Z}_{\geq 0}$ and $0 < t < 1$. For $D \subseteq \mathbb{R}^d$, the fractional order Sobolev space $H^s(D)$ consists of distributions v in D satisfying

$$\|v\|^2_{H^s(D)} := \|v\|^2_{H^m(D)} + \sum_{|\gamma|=m} \int_D \int_D \frac{|\partial^\gamma v(x) - \partial^\gamma v(y)|^2}{|x-y|^{d+2t}} dxdy < \infty.$$

Denote by $H^s_0(D)$ the closure of $C^\infty_0(D)$ in $H^s(D)$, and by $H^{-s}(D)$ the dual space of $H^s_0(D)$. For $s \geq -1$, $[H^s(\Omega)]^2$ represents the vector space, such that $\mathbf{v} = (v_1, v_2)^T \in [H^s(\Omega)]^2$ represents $v_i \in H^s(\Omega)$, $i = 1, 2$, where $(\cdot, \cdot)^T$ is the transposition of a matrix or a vector, and $\|\mathbf{v}\|^2_{[H^s(\Omega)]^2} := \|v_1\|^2_{H^s(\Omega)} + \|v_2\|^2_{H^s(\Omega)}$. We denote curl $\mathbf{v} := \partial_1 v_2 - \partial_2 v_1$. For a scalar function ψ, we denote **curl** $\psi := (\partial_2\psi, -\partial_1\psi)^T$.

By Green's formula, the variational formulation for the biharmonic problem (1.1) can be written as:

$$a(\phi, \psi) := \int_\Omega \Delta\phi\Delta\psi dx = \int_\Omega f\psi dx = (f, \psi) \quad \forall\psi \in H^2_0(\Omega). \tag{2.1}$$

For a function $\psi \in H^2_0(\Omega)$, applying the Poincaré-type inequality [33] twice, it follows

$$a(\psi, \psi) = \|\Delta\psi\|^2 \geq C|\psi|^2_{H^2(\Omega)} \geq C\|\psi\|^2_{H^2(\Omega)}.$$

Thus, for any $f \in H^{-2}(\Omega)$, by the Lax-Milgram Theorem (2.1) admits a unique solution $\phi \in H^2_0(\Omega)$.

The regularity of the solution ϕ depends on the given data f and the domain geometry [1, 10]. In order to decouple (1.1), we assume that the polygonal domain Ω consists of N vertices Q_i, $i = 1, \cdots, N$, and the corresponding interior angles are $\omega_i \in (0, 2\pi)\backslash\{\pi\}$. Let $\omega = \max_i \omega_i \in [\frac{\pi}{3}, 2\pi)\backslash\{\pi\}$ be the largest interior angle associated with the vertex Q. An example of the domain is given in Fig. 1. Let z_ℓ, $\ell = 1, 2, \ldots$ satisfying $\mathrm{Re}(z_\ell) > 0$ be the solutions of the following characteristic

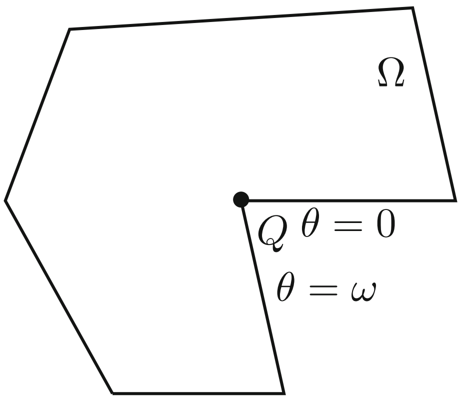


Fig. 1 Domain Ω containing one reentrant corner

equation

$$\sin^2(z\omega) = z^2 \sin^2(\omega). \tag{2.2}$$

Define a threshold value

$$\alpha_0 := \inf\{\mathrm{Re}(z_\ell),\ \ell = 1, 2, \ldots, \} > \frac{1}{2}. \tag{2.3}$$

Then according to the regularity estimate [6, 11, 33, 35], when $0 < \alpha < \alpha_0$, the solution of the biharmonic problem (1.1) satisfies

$$\|\phi\|_{H^{2+\alpha}(\Omega)} \le C\|f\|_{H^{-2+\alpha}(\Omega)}.$$

as long as ω satisfies

$$\sin\sqrt{\frac{\omega^2}{\sin\omega^2} - 1} \ne \pm\sqrt{1 - \frac{\sin\omega^2}{\omega^2}}. \tag{2.4}$$

The graph of α_0 in terms of ω is shown in Fig. 2(a), in which we also display the value

$$\beta_0 = \frac{\pi}{\omega}, \tag{2.5}$$

where β_0 is the threshold value of the regularity characteristic equation for the Poisson equation

$$-\Delta\psi = f \quad \text{in } \Omega, \qquad \psi = 0 \quad \text{on } \partial\Omega. \tag{2.6}$$

According to the regularity estimate [32, 33], when $0 < \beta < \beta_0$, the solution of the Poisson equation (2.6) satisfies

$$\|\psi\|_{H^{1+\beta}(\Omega)} \le C\|f\|_{H^{-1+\beta}(\Omega)}.$$

By observation, there exists an angle $\tilde{\omega} \approx 0.3548\pi$, such that when $\omega > \tilde{\omega}$ (resp. $\omega < \tilde{\omega}$), it holds $\alpha_0 < \beta_0 + 1$ (resp. $\alpha_0 > \beta_0 + 1$). We display $\alpha_0 - \beta_0$ in terms of ω in Fig. 2(b). In Table 1, we present some numerical values of α_0 for different interior angles ω.

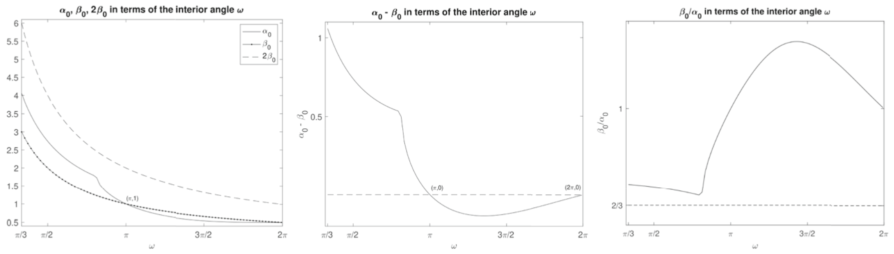
For α_0 and β_0 given above, we have the following result from [35, Theorem 7.1.1].

Lemma 2.1 *For $\beta_0 = \frac{\pi}{\omega}$, if $\omega \in (0, \pi)$, then α_0 in (2.3) satisfies*

$$\beta_0 < \alpha_0 < 2\beta_0; \tag{2.7}$$

Table 1 Values of α_0 for different interior angles ω

ω	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	$\frac{7\pi}{6}$	$\frac{6\pi}{5}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$
$\alpha_0 \approx$	4.0593	2.7396	2.0941	1.8854	1.5339	1.2006	0.7520	0.7178	0.6736	0.6157	0.5445	0.5050



(a) (b) (c)

Fig. 2 (a) α_0, β_0 in terms of ω; (b) $\alpha_0 - \beta_0$ in terms of ω; (c) β_0/α_0 in terms of ω

and if $\omega \in (\pi, 2\pi)$*, then it holds*

$$\frac{1}{2} < \alpha_0 < \beta_0. \tag{2.8}$$

In this paper, the parameters α and β play important roles in the regularity estimates, and in the rest of this paper we always assume

$$0 < \alpha < \alpha_0, \quad 0 < \beta < \beta_0. \tag{2.9}$$

2.2 The decoupled formulation of the biharmonic problem and the regularities

The decomposition of biharmonic equation (1.1) into the Poisson equations and the Stokes equation has been known for a long time, for example from the context of plate models [18, 26] or stream function formulations [30, 31]. However, the decomposition mainly consists of two Poisson equations and one Stokes equation. Here, we will introduce a new decomposition that involves only one Stokes equation and one Poisson equation. Subsequently, we analyze the regularity in the polygonal domain.

It is known that solving high order problems numerically, such as (1.1), is more difficult than solving the lower order problem. To decompose (1.1) into lower order problems, we first introduce a steady-state Stokes problem

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p =& \mathbf{F} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} =& 0 \quad \text{in } \Omega, \\ \mathbf{u} =& 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.10}$$

where $\mathbf{u} = (u_1, u_2)^T$ is the velocity field of an incompressible fluid motion and p is the associated pressure. Throughout the paper the source term $\mathbf{F} = (f_1, f_2)^T$ is assumed to satisfy

$$\operatorname{curl} \mathbf{F} = \partial_1 f_2 - \partial_2 f_1 = f, \tag{2.11}$$

where f is the given function in (1.1).

The weak formulation of the Stokes equation (2.10) is to find $\mathbf{u} \in [H_0^1(\Omega)]^2$ and $p \in L_0^2(\Omega)$ such that

$$\begin{aligned} (\nabla \mathbf{u}, \nabla \mathbf{v}) - (\operatorname{div} \mathbf{v}, p) =& \langle \mathbf{F}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in [H_0^1(\Omega)]^2, \\ -(\operatorname{div} \mathbf{u}, q) =& 0 \quad \forall q \in L_0^2(\Omega), \end{aligned} \tag{2.12}$$

where

$$L_0^2(\Omega) = \{q \in L^2(\Omega), \int_\Omega q dx = 0\}.$$

For the bilinear forms in (2.12), we have the following Ladyzhenskaya-Babuska-Breezi (LBB) or inf-sup conditions,

$$\begin{aligned} \inf_{q \in L_0^2(\Omega)} \sup_{\mathbf{v} \in [H_0^1(\Omega)]^2} \frac{-(\operatorname{div} \mathbf{v}, q)}{\|\mathbf{v}\|_{[H_0^1(\Omega)]^2} \|q\|} \geq C_1 > 0, \\ \inf_{\mathbf{u} \in [H_0^1(\Omega)]^2} \sup_{\mathbf{v} \in [H_0^1(\Omega)]^2} \frac{(\nabla \mathbf{u}, \nabla \mathbf{v})}{\|\mathbf{u}\|_{[H_0^1(\Omega)]^2} \|\mathbf{v}\|_{[H_0^1(\Omega)]^2}} \geq C_1 > 0, \end{aligned} \tag{2.13}$$

and the upper bounds

$$\begin{aligned} (\nabla \mathbf{u}, \nabla \mathbf{v}) &\leq C_2 \|\mathbf{u}\|_{[H_0^1(\Omega)]^2} \|\mathbf{v}\|_{[H_0^1(\Omega)]^2}, \\ (\operatorname{div} \mathbf{v}, q) &\leq C_2 \|\mathbf{v}\|_{[H_0^1(\Omega)]^2} \|q\|, \end{aligned} \tag{2.14}$$

where C_1, C_2 are some constants.

Given $\mathbf{F} \in [H^{-1}(\Omega)]^2$, under conditions (2.13) and (2.14), the weak formulation (2.12) admits a unique solution $(\mathbf{u}, p) \in [H_0^1(\Omega)]^2 \times L_0^2(\Omega)$ [30, 38, 54]. Moreover, if $\mathbf{F} \in [H^{-1+\alpha}(\Omega)]^2$ for $\alpha < \alpha_0$, the Stokes problem holds the regularity estimate [9, 33, 50],

$$\|\mathbf{u}\|_{[H^{1+\alpha}(\Omega)]^2} + \|p\|_{H^\alpha(\Omega)} \leq C \|\mathbf{F}\|_{[H^{-1+\alpha}(\Omega)]^2}. \tag{2.15}$$

For f in (1.1), we introduce a Poisson problem

$$-\Delta w = f \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega. \tag{2.16}$$

Then we have the following result.

Lemma 2.2 *For $f \in H^{-1}(\Omega)$, assume that w is the solution of (2.16). Then*

$$\mathbf{F} = \mathbf{curl}\, w \in [L^2(\Omega)]^2 \tag{2.17}$$

satisfies (2.11) and

$$\|\mathbf{F}\|_{[L^2(\Omega)]^2} \leq C \|f\|_{H^{-1}(\Omega)}. \tag{2.18}$$

Proof It is easy to verify that $\mathbf{F}$ satisfies (2.11), namely,

$$\text{curl } \mathbf{F} = \text{curl } (\mathbf{curl}\ w) = -\Delta w = f.$$

The Poisson problem (2.16) admits a unique $w \in H_0^1(\Omega)$, which satisfies $\|w\|_{H^1(\Omega)} \leq C\|f\|_{H^{-1}(\Omega)}$. Note that $\|\mathbf{curl}\ w\|_{[L^2(\Omega)]^2} = |w|_{H^1(\Omega)}$. Therefore, we have

$$\|\mathbf{curl}\ w\|_{[L^2(\Omega)]^2} \leq C\|f\|_{H^{-1}(\Omega)}. \tag{2.19}$$

Thus, the estimate (2.18) holds. □

In a polygonal domain Ω, if $f \in H^l(\Omega)$ for $l \geq -1$, the regularity estimate [32, 33, 39] for the Poisson problem (2.16) gives rise to

$$\|w\|_{H^{\min\{1+\beta,l+2\}}(\Omega)} \leq \|f\|_{H^l(\Omega)}, \tag{2.20}$$

where $\beta < \beta_0 = \frac{\pi}{\omega}$ given in (2.5) with ω being the largest interior angle of Ω. Thus for $\mathbf{F}$ defined in (2.17), we have $\mathbf{F} \in [H^{\min\{\beta,l+1\}}(\Omega)]^2$.

Note that $\mathbf{F}$ satisfying (2.11) is not unique. Assume that $\mathbf{F}_0 \in [L^2(\Omega)]^2$ is a solution of (2.11). Then

$$\mathbf{F} = \mathbf{F}_0 + \nabla q \quad \forall q \in H^1(\Omega),$$

is also a solution of (2.11) in $[L^2(\Omega)]^2$, since for $q \in H^1(\Omega)$, it holds (curl $\nabla q) \equiv 0$.

In addition to Lemma 2.2, the source term $\mathbf{F}$ could be explicitly derived in some special cases.

Remark 2.3 Assume that $f \in L^2(\Omega)$. If $f(\xi, x_2)$ and $f(x_1, \zeta)$ are integrable in terms of ξ and η, then for any constant η, the source term

$$\mathbf{F} = \left[-\eta \int_{c_2}^{x_2} f(x_1, \zeta)d\zeta,\ (1-\eta)\int_{c_1}^{x_1} f(\xi, x_2)d\xi\right]^T \in [L^2(\Omega)]^2 \tag{2.21}$$

satisfies (2.11), where c_1 and c_2 are any constants that are suitable for the integrals above.

For all these $\mathbf{F} \in [L^2(\Omega)]^2$ satisfying (2.11), we have the following result.

Theorem 2.4 *Assume that $\mathbf{F}_l \in [L^2(\Omega)]^2 \cap [H^{\alpha-1}(\Omega)]^2$, $l = 1, 2$ both satisfy (2.11). Let $(\mathbf{u}_l, p_l)$ be solutions of (2.10) or (2.12) corresponding to $\mathbf{F}_l$. Then it follows that*

$$\begin{aligned} \mathbf{u}_1 =& \mathbf{u}_2 \quad \text{in } [H_0^1(\Omega)]^2 \cap [H^{1+\alpha}(\Omega)]^2, \\ p_1 =& p_2 + q \quad \text{in } L_0^2(\Omega) \cap H^\alpha(\Omega), \end{aligned} \tag{2.22}$$

where $q \in L_0^2(\Omega) \cap H^1(\Omega)$ satisfies $\nabla q = \mathbf{F}_1 - \mathbf{F}_2$.

Proof We take $\bar{\mathbf{F}} = \mathbf{F}_1 - \mathbf{F}_2 \in [L^2(\Omega)]^2$, then by Helmholtz decomposition [30], there exist a stream-function ψ and a potential-function $q \in H^1(\Omega)$ uniquely up to a constant such that

$$\bar{\mathbf{F}} = \nabla q + \mathbf{curl}\ \psi, \tag{2.23}$$

and

$$(\bar{\mathbf{F}} - \nabla q) \cdot \mathbf{n} = (\mathbf{curl}\ \psi) \cdot \mathbf{n} = 0 \quad \text{in } H^{-\frac{1}{2}}(\partial\Omega). \tag{2.24}$$

From (2.24), we have

$$\partial_\tau \psi = (\mathbf{curl}\ \psi) \cdot \mathbf{n} = 0 \quad \text{in } H^{-\frac{1}{2}}(\partial\Omega),$$

where τ is the unit tangential vector on $\partial\Omega$. Thus we have

$$\psi = C_0 \quad \text{in } H^{\frac{1}{2}}(\partial\Omega), \tag{2.25}$$

where C_0 is a constant. Taking curl on (2.23) gives

$$-\Delta\psi = \text{curl}\ (\mathbf{curl}\ \psi) = \text{curl}\ \bar{\mathbf{F}} = 0, \tag{2.26}$$

where the last equality is based on the fact that $\mathbf{F}_1, \mathbf{F}_2$ satisfy (2.11). By the Lax-Milgram Theorem, the Poisson equation (2.26) with the boundary condition (2.25) admits a unique solution $\psi = C_0$ in $H^1(\Omega)$. Therefore, the decomposition (2.23) is equivalent to

$$\bar{\mathbf{F}} = \nabla q. \tag{2.27}$$

Let $\bar{\mathbf{u}} = \mathbf{u}_1 - \mathbf{u}_2$ and $\bar{p} = p_1 - p_2$, then $(\bar{\mathbf{u}}, \bar{p})$ satisfies

$$\begin{aligned} -\Delta\bar{\mathbf{u}} + \nabla(\bar{p} - q) &= 0 \quad \text{in } \Omega, \\ \text{div}\ \bar{\mathbf{u}} &= 0 \quad \text{in } \Omega, \\ \bar{\mathbf{u}} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.28}$$

By the regularity of the Stokes problem (2.28), the conclusion holds. □

Lemma 2.5 *Assume that the source term* $\mathbf{F} \in [L^2(\Omega)]^2 \cap [H^{-1+\alpha}(\Omega)]^2$ *of the Stokes problem* (2.10) *is any vector function determined by* $f \in H^{-1}(\Omega) \cap H^{-2+\alpha}(\Omega)$ *satisfying* (2.11). *Then* (2.10) *admits a unique solution* $\mathbf{u} \in [H^{1+\alpha}(\Omega)]^2$ *and satisfies*

$$\|\mathbf{u}\|_{[H^{1+\alpha}(\Omega)]^2} \leq C\|f\|_{H^{-1}(\Omega)}. \tag{2.29}$$

Proof Given $f \in H^{-1}(\Omega) \cap H^{-2+\alpha}(\Omega)$, we can always find a vector function $\mathbf{F}_0 \in [L^2(\Omega)]^2 \cap [H^{-1+\alpha}(\Omega)]^2$ following Lemma 2.2 such that the corresponding Stokes problem (2.10) admits a unique solution $\mathbf{u}_0 \in [H^{\max\{2,1+\alpha\}}(\Omega)]^2$ satisfying

$$\|\mathbf{u}_0\|_{[H^{1+\alpha}(\Omega)]^2} \leq C\|\mathbf{F}_0\|_{[H^{\max\{0,-1+\alpha\}}(\Omega)]^2} \leq C\|f\|_{H^{-1}(\Omega)}. \tag{2.30}$$

For any source term $\mathbf{F}$ also satisfying (2.17), it follows by Theorem 2.4 that the corresponding solution $\mathbf{u} = \mathbf{u}_0$, so the conclusion holds. □

To show the connection of the solution $\mathbf{u}$ to the Stokes problem (2.10) with the biharmonic problem (1.1), we introduce the following result from [31, Theorem 3.1].

Lemma 2.6 *A function $\mathbf{v} \in [H^m(\Omega)]^2$ for integer $m \geq 0$ satisfies*

$$\operatorname{div} \mathbf{v} = 0, \quad \langle \mathbf{v} \cdot \mathbf{n}, 1\rangle|_{\partial\Omega} = 0,$$

then there exists a stream function $\psi \in H^{m+1}(\Omega)$ uniquely up to an additive constant satisfying

$$\mathbf{v} = \mathbf{curl}\ \psi.$$

Since $\mathbf{u} \in [H_0^1(\Omega)]^2 \cap [H^{1+\alpha}(\Omega)]^2$ and $\operatorname{div} \mathbf{u} = 0$, we have by Lemma 2.6 that there exists $\bar{\phi} \in H^2(\Omega)$ uniquely up to an additive constant satisfying

$$(u_1, u_2)^T = \mathbf{u} = \mathbf{curl}\ \bar{\phi} = (\partial_2\bar{\phi}, -\partial_1\bar{\phi})^T, \tag{2.31}$$

which further implies $|\nabla\bar{\phi}| \in H^{1+\alpha}(\Omega)$, thus we have

$$\bar{\phi} \in H^{2+\alpha}(\Omega). \tag{2.32}$$

Lemma 2.7 *There exists a unique $\bar{\phi} \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega)$ satisfying (2.31).*

Proof By definition, $\partial_\tau\bar{\phi} = \mathbf{curl}\ \bar{\phi} \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n} = 0$, where τ is the unit tangent to $\partial\Omega$. Thus it follows

$$\bar{\phi} = \text{constant} \quad \text{on } \partial\Omega.$$

Without loss of generality, we can take

$$\bar{\phi} = 0 \quad \text{on } \partial\Omega. \tag{2.33}$$

From (2.31), we also have

$$\nabla\bar{\phi} = (\partial_1\bar{\phi}, \partial_2\bar{\phi})^T = (-u_2, u_1)^T = \mathbf{0} \quad \text{on } \partial\Omega. \tag{2.34}$$

Thus, the conclusion follows from (2.33), (2.34), (2.32), and Lemma 2.6. □

Instead of solving for $\bar{\phi} \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega)$ from (2.31) directly, we apply the curl operator on (2.31) to obtain the following Poisson problem

$$-\Delta\bar{\phi} = \operatorname{curl} \mathbf{u} \quad \text{in } \Omega \quad \text{and} \quad \bar{\phi} = 0 \quad \text{on } \partial\Omega. \tag{2.35}$$

The weak formulation of (2.35) is to find $\bar{\phi} \in H_0^1(\Omega)$, such that

$$(\nabla\bar{\phi}, \nabla\psi) = (\text{curl}\,\mathbf{u}, \psi) \quad \forall\psi \in H_0^1(\Omega). \tag{2.36}$$

Since $\text{curl}\,\mathbf{u} \in L^2(\Omega)$, we have by the Lax-Milgram Theorem that (2.36) admits a unique solution $\bar{\phi} \in H_0^1(\Omega)$.

Lemma 2.8 *The Poisson problem (2.35) admits a unique solution $\bar{\phi} \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega)$.*

Proof Since $\bar{\phi} \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega) \subset H_0^1(\Omega)$ is a solution of (2.31), so it is also a solution of the Poisson problem (2.35). By the uniqueness of the solution of (2.35) in $H_0^1(\Omega)$, the conclusion holds. □

Lemma 2.9 *The solution $\bar{\phi} \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega)$ obtained through the Poisson problem (2.35) satisfies the biharmonic problem*

$$\Delta^2\bar{\phi} = \text{curl}\,\mathbf{F} = f \quad \textit{in} \;\; \Omega, \qquad \textit{and} \quad \bar{\phi} = 0 \qquad \partial_{\mathbf{n}}\bar{\phi} = 0 \quad \textit{on} \; \partial\Omega. \tag{2.37}$$

Proof Following (2.31), we replace $\mathbf{u}$ by $\mathbf{curl}\,\bar{\phi}$ in (2.10) and obtain

$$-\Delta(\partial_2\bar{\phi}) + \partial_1 p = f_1 \quad \text{in } \Omega, \tag{2.38a}$$

$$-\Delta(-\partial_1\bar{\phi}) + \partial_2 p = f_2 \quad \text{in } \Omega. \tag{2.38b}$$

Applying differential operators $-\partial_2$ and ∂_1 to (2.38a) and (2.38b), respectively, and taking the summation leads to the conclusion. □

From Lemma 2.9, we find that $\bar{\phi}$ in (2.37) satisfies exactly the same problem as ϕ in (1.1) in the following sense,

$$\phi = \bar{\phi} \quad \text{in } H_0^2(\Omega) \cap H^{2+\alpha}(\Omega). \tag{2.39}$$

From now on, we use ϕ to replace the notation $\bar{\phi}$. Thus the Poisson problem (2.35) is equivalent to

$$-\Delta\phi = \text{curl}\,\mathbf{u} \quad \text{in } \Omega, \quad \phi = 0 \quad \text{on } \partial\Omega. \tag{2.40}$$

The weak formulation of (2.40) is to find $\phi \in H_0^1(\Omega)$ satisfying

$$(\nabla\phi, \nabla\psi) = (\text{curl}\,\mathbf{u}, \psi) \quad \forall\psi \in H_0^1(\Omega). \tag{2.41}$$

By the regularity in Lemma 2.5 and Lemma 2.7, we have that

$$\|\phi\|_{H^{2+\alpha}(\Omega)} \le C\|\text{curl}\,\mathbf{u}\|_{H^\alpha(\Omega)} \le C\|f\|_{H^{\max\{-1,\alpha-2\}}(\Omega)}. \tag{2.42}$$

Based on the discussion above, we conclude the main result in this section.

Theorem 2.10 *(S-P decomposition) Given $f \in H^{-1}(\Omega)$ and an appropriate $\mathbf{F} \in [L^2(\Omega)]^2$ satisfying (2.11), the solution of the biharmonic problem with Dirichlet boundary conditions (1.1) is equivalent to that of the following S-P decomposition:*

(a) *Solve $\mathbf{u}$ from the Stokes problem (2.10);*
(b) *Solve ϕ from the Poisson problem (2.40).*

For the S-P decomposition, Theorem 2.4 implies that there are no restrictions on how $\mathbf{F} \in [L^2(\Omega)]^2$ is obtained. It can be obtained by a direct calculation as in (2.3) or by solving a Poisson problem with a non-homogenous boundary condition that can produce a higher regularity $\mathbf{F}$. Especially, if $\mathbf{F}$ is obtained by solving the Poisson problem (2.16), the S-P decomposition in Theorem 2.10 reduces to the well-known P-S-P decomposition [18, 19].

Corollary 2.11 *(P-S-P decomposition) Given $f \in H^{-1}(\Omega)$, the solution of the biharmonic problem with Dirichlet boundary conditions (1.1) can be recovered by the following P-S-P decomposition:*

(a) *Solve the Poisson problem (2.16), and obtain $\mathbf{F}$ following* Lemma 2.2*;*
(b) *Solve $\mathbf{u}$ from the Stokes problem (2.10);*
(c) *Solve ϕ from the Poisson problem (2.40).*

Remark 2.12 Note that our analysis does not directly extend to the three-dimensional (3D) problem because the identity $\Delta = \operatorname{curl} \mathbf{curl}$, which is essential for deriving the decomposition in 2D, does not hold in 3D. However, it is possible to develop similar strategies in 3D by imposing additional conditions on the pressure term. We plan to investigate the 3D case in a forthcoming project.

Remark 2.13 We mention that the biharmonic problem investigated in [45] has the Navier boundary conditions:

$$\Delta^2\phi = f \quad \text{in } \Omega, \qquad \phi = \Delta\phi = 0 \quad \text{on } \partial\Omega, \tag{2.43}$$

which is different from the Dirichlet boundary conditions we study in this paper. In contrast to the Dirichlet boundary conditions in problem (1.1), the Navier boundary conditions allow the biharmonic problem (2.43) to be completely decoupled into two Poisson problems. The decomposition for problem (1.1) is different in nature to incorporate the Dirichlet boundary conditions. Another difference is that the decomposition for problem (2.43) involves a third Poisson equation in nonconvex domains to confine the solution in the correct function space. On the contrary, this is not a concern for our method solving the Dirichlet problem.

3 The finite element method and error estimates

In this section, we propose a C^0 finite element method for solving the biharmonic problem (1.1) based on the S-P decomposition and the P-S-P decomposition.

3.1 The finite element algorithm

Let $\mathcal{T}_n$ be a triangulation of Ω with shape-regular triangles and let $\mathcal{P}_k(\mathcal{T}_n)$ be the C^0 Lagrange finite element space associated with $\mathcal{T}_n$,

$$\mathcal{P}_k(\mathcal{T}_n) := \{v \in C^0(\Omega) : \ v|_T \in P_k, \ \forall T \in \mathcal{T}_n\}, \tag{3.1}$$

where P_k is the space of polynomials of degree no more than k. Further, we introduce the following specific C^0 Lagrange finite element spaces associated with $\mathcal{T}_n$,

$$\begin{aligned} V_n^k :=& \mathcal{P}_k(\mathcal{T}_n) \cap H_0^1(\Omega), \\ S_n^k :=& \mathcal{P}_k(\mathcal{T}_n) \cap L_0^2(\Omega), \end{aligned} \tag{3.2}$$

and the bubble function space

$$B_n^3 := \{v \in C^0(\Omega) : \ v|_T \in \operatorname{span}\{\lambda_1\lambda_2\lambda_3\}, \ \forall T \in \mathcal{T}_n\},$$

where λ_i, $i = 1, 2, 3$ are the barycentric coordinates on T.

We introduce the finite element algorithm for the biharmonic problem (1.1) by utilizing the P-S-P decomposition as follows.

Algorithm 3.1 (P-S-P finite element algorithm) For any $f \in H^{-1}(\Omega)$ and $k \geq 1$, we consider the following steps.

- Step 1. Find $w_n \in V_n^k$ in the Poisson equation

$$(\nabla w_n, \nabla \psi) = (f, \psi) \quad \forall \psi \in V_n^k. \tag{3.3}$$

  Then define $\mathbf{F}_n = \mathbf{curl}\ w_n$.
- Step 2. If $k = 1$, we find the Mini element approximation $\mathbf{u}_n \times p_n \in [V_n^1 \oplus B_n^3]^2 \times S_n^1$ of the Stokes equation

$$\begin{aligned} (\nabla \mathbf{u}_n, \nabla \mathbf{v}) - (p_n, \operatorname{div} \mathbf{v}) =& \langle \mathbf{F}_n, \mathbf{v}\rangle \quad \forall \mathbf{v} \in [V_n^1 \oplus B_n^3]^2, \\ -(\operatorname{div} \mathbf{u}_n, q) =& 0 \quad \forall q \in S_n^1. \end{aligned} \tag{3.4}$$

  If $k \geq 2$, we find the Taylor-Hood element solution $\mathbf{u}_n \times p_n \in [V_n^k]^2 \times S_n^{k-1}$ of the Stokes equation

$$\begin{aligned} (\nabla \mathbf{u}_n, \nabla \mathbf{v}) - (p_n, \operatorname{div} \mathbf{v}) =& \langle \mathbf{F}_n, \mathbf{v}\rangle \quad \forall \mathbf{v} \in [V_n^k]^2, \\ -(\operatorname{div} \mathbf{u}_n, q) =& 0 \quad \forall q \in S_n^{k-1}. \end{aligned} \tag{3.5}$$

- Step 3. Find the finite element solution $\phi_n \in V_n^k$ of the Poisson equation

$$(\nabla \phi_n, \nabla \psi) = (\operatorname{curl} \mathbf{u}_n, \psi) \quad \forall \psi \in V_n^k. \tag{3.6}$$

More generally, Algorithm 3.1 can be updated based on the S-P decomposition as follows.

Algorithm 3.2 (S-P finite element algorithm) Given $f \in L^2(\Omega)$ and an appropriate $\mathbf{F} \in [L^2(\Omega)]^2$ satisfying (2.11), as in (2.21), we have a simpler version of the algorithm.

- Step 1. Same as Algorithm 3.1 Step 2 with $\langle \mathbf{F}_n, \mathbf{v} \rangle$ replaced by $\langle \mathbf{F}, \mathbf{v} \rangle$.
- Step 2. Same as Algorithm 3.1 Step 3.

The finite element approximations for the Poisson problems in both Algorithms 3.1 and 3.2 are well defined by the Lax-Milgram Theorem. We take the Mini element method [4] and the Taylor-Hood element method [17, 55] for solving the Stokes problem, other finite element methods could also be used.

Remark 3.3 Both Algorithms 3.1 and 3.2 are solved serially, but each of them can also be solved as a coupled system. However, solving it serially is more cost-effective.

3.2 Optimal error estimates on quasi-uniform meshes

For error analysis purposes, we suppose that the mesh $\mathcal{T}_n$ consists of quasi-uniform triangles with size h. We mention that the methods in [18, 19, 21, 28] are also based on the P-S-P decomposition. By assuming that $w \in H^{1+s}(\Omega)$, $\mathbf{u} \in [H^{1+s}(\Omega)]^2$, $p \in H^s(\Omega)$, and $\phi \in H^{2+s}(\Omega)$, $0 < s \leq 1$, the following error estimates can be obtained for Algorithm 3.1 based on the standard error analysis [21, 28]

$$\begin{aligned} \|w - w_n\|_{H^1(\Omega)} + \|\mathbf{u} - \mathbf{u}_n\|_{[H^1(\Omega)]^2} + \|p - p_n\| + \|\phi - \phi_n\|_{H^1(\Omega)} &\leq Ch^s, \\ \|w - w_n\| + \|\mathbf{u} - \mathbf{u}_n\|_{[L^2(\Omega)]^2} + \|\phi - \phi_n\| &\leq Ch^{2s}. \end{aligned} \tag{3.7}$$

In the following, we conduct an error analysis of Algorithms 3.1 and 3.2 for the biharmonic problem (1.1) in both convex and non-convex polygonal domains, utilizing P_k polynomials. This analysis is grounded on the regularity estimates outlined in Section 2, which characterize the regularity of the solution $w \in H^{1+\beta}(\Omega)$, $\mathbf{u} \in [H^{1+\alpha}(\Omega)]^2$, $p \in H^\alpha(\Omega)$, and $\phi \in H^{2+\alpha}(\Omega)$, where $\alpha \in (0, \alpha_0)$ and $\beta \in (0, \beta_0)$ with α_0 in (2.3) and β_0 in (2.5) depending on the largest interior angle ω of the polygonal domain. Specially, when the largest interior angle of the domain $\omega \in (\pi, 2\pi)$, it holds $0 < \alpha < \beta < 1$. In such cases, we employ new techniques to achieve higher convergence rates for certain approximations compared to the estimates (3.7). A summary of such a case will be provided at the end of this subsection.

The interpolation error estimate on $\mathcal{T}_n$ (see e.g., [15]) for any $v \in H^\sigma(\Omega) \cap H_0^1(\Omega)$, $\sigma > 1$,

$$\|v - v_I\|_{H^l(\Omega)} \leq Ch^{\sigma - l}\|v\|_{H^\sigma(\Omega)}, \tag{3.8}$$

where $l = 0, 1$ and $v_I \in V_n^k$ represents the nodal interpolation of v.

To make the analysis simple and clear, we assume that $f \in H^{\max\{\alpha_0, \beta_0\}-1}(\Omega) \cap L^2(\Omega)$, where α_0, β_0 are given in (2.3) and (2.5), respectively. Since Algorithm 3.2

involves only one Stokes equation and one Poisson equation, we first give the error estimate of Algorithm 3.2.

For $f \in H^{\max\{\alpha_0,\beta_0\}-1}(\Omega) \cap L^2(\Omega)$, if $\mathbf{F}$ is explicitly calculated, as in (2.21), we have $\mathbf{u} \in [H^{1+\alpha}(\Omega)]^2$, $p \in H^{\alpha}(\Omega)$. Note that the bilinear forms in the Mini element method ($k = 1$) or Taylor-Hood element method ($k \geq 2$) satisfying the LBB condition on quasi-uniform meshes [4, 17, 55]. Then the standard arguments for error estimate (see e.g., [13, 31, 52]) give the following error estimates.

Lemma 3.4 *Let* $(\mathbf{u}, p)$ *be the solution of the Stokes problem* (2.12)*, and* $(\mathbf{u}_n, p_n)$ *be the Mini element solution* ($k = 1$) *or Taylor-Hood element solution* ($k \geq 2$) *in* Algorithm 3.2 *on quasi-uniform meshes, then it follows*

$$\|\mathbf{u} - \mathbf{u}_n\|_{[H^1(\Omega)]^2} + \|p - p_n\| \leq Ch^{\min\{k,\alpha\}} D_2, \tag{3.9a}$$

$$\|\mathbf{u} - \mathbf{u}_n\|_{[L^2(\Omega)]^2} \leq Ch^{\min\{k+1,\alpha+1,2\alpha\}} D_2, \tag{3.9b}$$

$$\|\mathbf{u} - \mathbf{u}_n\|_{[H^{-1}(\Omega)]^2} \leq Ch^{\min\{2k,k+2,k+\alpha,\alpha+2,2\alpha\}} D_2, \tag{3.9c}$$

where $D_2 = \|\mathbf{u}\|_{[H^{\min\{k+1,\alpha+1\}}(\Omega)]^2} + \|p\|_{H^{\min\{k,\alpha\}}(\Omega)}$.

Remark 3.5 If the largest interior angle $\omega < \pi$, it follows $\min\{\alpha + 1, 2\alpha\} = \alpha + 1$, and if $\omega > \pi$, we have $\min\{\alpha + 1, 2\alpha\} = 2\alpha$.

If $\mathbf{F}$ is given by Lemma 2.2, then for the Poisson equation (2.16) in a polygonal domain with $f \in H^{\max\{\alpha_0,\beta_0\}-1}(\Omega) \cap L^2(\Omega)$, the regularity estimate gives $w \in H^{1+\beta}(\Omega)$ for $\beta < \beta_0 = \frac{\pi}{\omega}$ (see, e.g., [32, 33]). Therefore, we have

$$\mathbf{F} = \mathbf{curl}\ w \in [H^{\beta}(\Omega)]^2 \subset [H^{\min\{\alpha-1,\beta\}}(\Omega)]^2,$$

where $\alpha < \alpha_0$, and we also have $\mathbf{u} \in [H^{\min\{\alpha+1,\beta+2\}}(\Omega)]^2$, $p \in H^{\min\{\alpha,\beta+1\}}(\Omega)$, and $\phi \in H^{\min\{\alpha+2,\beta+3\}}(\Omega)$. For the finite element approximations w_n in (3.3), the standard error estimate [22] yields

$$\|w - w_n\|_{H^1(\Omega)} \leq Ch^{\min\{k,\beta\}} \|w\|_{H^{\min\{k+1,\beta+1\}}}, \quad \|w - w_n\| \leq Ch^{\min\{k+1,\beta+1,2\beta\}} \|w\|_{H^{\min\{k+1,\beta+1\}}} \tag{3.10}$$

which implies that

$$\begin{aligned} \|\mathbf{F} - \mathbf{F}_n\|_{[L^2(\Omega)]^2} &= \|\mathbf{curl}\ w - \mathbf{curl}\ w_n\|_{[L^2(\Omega)]^2} \leq \|w - w_n\|_{H^1(\Omega)} \leq Ch^{\min\{k,\beta\}} \|w\|_{H^{\min\{k+1,\beta+1\}}(\Omega)}, \\ \|\mathbf{F} - \mathbf{F}_n\|_{[H^{-1}(\Omega)]^2} &= \|\mathbf{curl}\ w - \mathbf{curl}\ w_n\|_{[H^{-1}(\Omega)]^2} \leq C\|w - w_n\| \leq Ch^{\min\{k+1,\beta+1,2\beta\}} \|w\|_{H^{\min\{k+1,\beta+1\}}(\Omega)}. \end{aligned} \tag{3.11}$$

For $\mathbf{F}_n$ in Algorithm 3.1, we further have the following result.

Lemma 3.6 *If* $\mathbf{F}_n$ *is given by* $\mathbf{F}_n = \mathbf{curl}\ w_n$ *in Step 1 of* Algorithm 3.1*, and* $\mathbf{F} = \mathbf{curl}\ w$ *is given in* (2.17)*, then it follows*

$$\langle \mathbf{F} - \mathbf{F}_n, \mathbf{curl}\ \psi \rangle = 0 \quad \forall \psi \in V_n^k. \tag{3.12}$$

Proof Subtracting (3.3) from the weak formulation of (2.16) gives the Galerkin orthogonality,

$$(\nabla(w-w_n),\nabla\psi)=(\partial_1 w-\partial_1 w_n,\partial_1\psi)+(\partial_2 w-\partial_2 w_n,\partial_2\psi)=0\quad\forall\psi\in V_n^k, \tag{3.13}$$

which implies that

$$\langle \mathbf{F}-\mathbf{F}_n,\mathbf{curl}\,\psi\rangle=\langle\mathbf{curl}\,(w-w_n),\mathbf{curl}\,\psi\rangle=(\partial_2 w-\partial_2 w_n,\partial_2\psi)+(\partial_1 w-\partial_1 w_n,\partial_1\psi)=0. \tag{3.14}$$

□

Next, we consider the error estimates of Taylor-Hood element approximations. Subtracting (3.5) from (2.12) gives

$$(\nabla(\mathbf{u}-\mathbf{u}_n),\nabla\mathbf{v})-(\mathrm{div}\,\mathbf{v},p-p_n)=\langle\mathbf{F}-\mathbf{F}_n,\mathbf{v}\rangle\quad\forall\mathbf{v}\in[V_n^k]^2, \tag{3.15a}$$
$$-(\mathrm{div}\,(\mathbf{u}-\mathbf{u}_n),q)=0\quad\forall q\in S_n^{k-1}. \tag{3.15b}$$

If $\mathbf{F}_n$ is calculated exactly or the L^2 projection of $\mathbf{F}$, it holds the estimates [16]

$$\|\mathbf{u}-\mathbf{u}_n\|_{[H^1(\Omega)]^2}+\|p-p_n\|\le C\left(\inf_{\mathbf{v}\in[V_n^k]^2}\|\mathbf{u}-\mathbf{v}\|_{[H^1(\Omega)]^2}+\inf_{q\in S_n^{k-1}}\|p-q\|\right), \tag{3.16}$$

otherwise applying Strang's first Lemma gives

$$\|\mathbf{u}-\mathbf{u}_n\|_{[H^1(\Omega)]^2}+\|p-p_n\|\le C\left(\inf_{\mathbf{v}\in[V_n^k]^2}\|\mathbf{u}-\mathbf{v}\|_{[H^1(\Omega)]^2}+\inf_{q\in S_n^{k-1}}\|p-q\|+\|\mathbf{F}-\mathbf{F}_n\|_{[H^{-1}(\Omega)]^2}\right). \tag{3.17}$$

To obtain the error estimates, we further introduce the adjoint problem of the Stokes equations (2.10),

$$\begin{aligned}-\Delta\mathbf{r}+\nabla s&=\mathbf{g}\quad\text{in }\Omega,\\ \mathrm{div}\,\mathbf{r}&=0\quad\text{in }\Omega,\\ \mathbf{r}&=0\quad\text{on }\partial\Omega,\end{aligned} \tag{3.18}$$

where $\mathbf{g}\in[H_0^l(\Omega)]^2$ for some $l=0,1$. Here, the notation $H_0^0(\Omega):=H^0(\Omega)=L^2(\Omega)$. The weak formulation of (3.18) is to find $\mathbf{r}\in[H_0^1(\Omega)]^2$ and $s\in L_0^2(\Omega)$ such that

$$(\nabla\mathbf{r},\nabla\mathbf{v})-(\mathrm{div}\,\mathbf{v},s)=\langle\mathbf{g},\mathbf{v}\rangle\quad\forall\mathbf{v}\in[H_0^1(\Omega)]^2, \tag{3.19a}$$
$$-(\mathrm{div}\,\mathbf{r},q)=0\quad\forall q\in L_0^2(\Omega). \tag{3.19b}$$

We have the following regularity result [9, 33, 50],

$$\|\mathbf{r}\|_{[H^{1+\min\{\alpha,l+1\}}(\Omega)]^2} + \|s\|_{H^{\min\{\alpha,l+1\}}(\Omega)} \le C\|\mathbf{g}\|_{[H^{\min\{\alpha,l+1\}-1}(\Omega)]^2} \le C\|\mathbf{g}\|_{[H^l(\Omega)]^2}, \quad (3.20)$$

where $\alpha < \alpha_0$.

Note that $\mathbf{r} \in [H^{1+\min\{\alpha,l+1\}}(\Omega)]^2$ satisfying (3.20) and (3.18). By Lemma 2.6 and Lemma 2.7, there exists $\psi \in H^{2+\min\{\alpha,l+1\}}(\Omega) \cap H_0^1(\Omega)$ such that

$$\mathbf{r} = \mathbf{curl}\ \psi. \quad (3.21)$$

We also have that $\|\psi\|_{H^{2+\min\{\alpha,l+1\}}(\Omega)} \le C\|\mathbf{r}\|_{[H^{1+\min\{\alpha,l+1\}}(\Omega)]^2}$.

Let $(\mathbf{r}_n, s_n)$ be the Taylor-Hood solution ($k \ge 2$) of (3.18). Then we have

$$(\nabla \mathbf{r}_n, \nabla \mathbf{v}) - (\operatorname{div} \mathbf{v}, s_n) = \langle \mathbf{g}, \mathbf{v}\rangle \quad \forall \mathbf{v} \in [V_n^k]^2, \quad (3.22a)$$

$$-(\operatorname{div} \mathbf{r}_n, q) = 0 \quad \forall q \in S_n^{k-1}. \quad (3.22b)$$

Subtracting (3.22) from (3.19) gives

$$(\nabla(\mathbf{r} - \mathbf{r}_n), \nabla \mathbf{v}) - (\operatorname{div} \mathbf{v}, s - s_n) = 0 \quad \forall \mathbf{v} \in [V_n^k]^2, \quad (3.23a)$$

$$-(\operatorname{div} (\mathbf{r} - \mathbf{r}_n), q) = 0 \quad \forall q \in S_n^{k-1}. \quad (3.23b)$$

Then we have the following error estimates.

Lemma 3.7 *Let $f \in H^{\max\{\alpha_0,\beta_0\}-1}(\Omega) \cap L^2(\Omega)$ for α_0, β_0 given in (2.3) and (2.5), respectively. Let $(\mathbf{u}, p)$ be the solution of the Stokes problem (2.12), and $(\mathbf{u}_n, p_n)$ be the Mini element solution ($k = 1$) or Taylor-Hood element solution ($k \ge 2$) in* Algorithm 3.1 *on quasi-uniform meshes. Then it follows the error estimates*

$$\|\mathbf{u} - \mathbf{u}_n\|_{[H^1(\Omega)]^2} + \|p - p_n\| \le Ch^{\min\{k,\alpha,\beta+1\}} D_1, \quad (3.24a)$$

$$\|\mathbf{u} - \mathbf{u}_n\|_{[L^2(\Omega)]^2} \le Ch^{\min\{k+1,\alpha+1,\beta+2,2\alpha\}} D_1, \quad (3.24b)$$

$$\|\mathbf{u} - \mathbf{u}_n\|_{[H^{-1}(\Omega)]^2} \le Ch^{\min\{2k,k+2,k+\beta,\alpha+2,\beta+3,2\alpha\}} D_1, \quad (3.24c)$$

where $D_1 = \|\mathbf{u}\|_{[H^{\min\{k+1,\alpha+1,\beta+2\}}(\Omega)]^2} + \|p\|_{H^{\min\{k,\alpha,\beta+2\}}(\Omega)} + \|w\|_{H^{\min\{k+1,\beta+1\}}(\Omega)}$.

The proof is given in Appendix A.1.

In the rest of this section, we follow the notations D_2 and D_1 defined in Lemmas 3.4 and 3.7, respectively. Then the following results hold.

Theorem 3.8 *Let $\phi_n \in V_n^k$ be the finite element solution of (3.6) from* Algorithm 3.2 *or* Algorithm 3.1*, and ϕ be the solution of the biharmonic problem (1.1). If ϕ_n is the solution of* Algorithm 3.2*,*

$$\|\phi - \phi_n\|_{H^1(\Omega)} \le Ch^{\min\{k,\alpha+1,2\alpha\}}(D_2 + \|\phi\|_{H^{\min\{k+1,\alpha+2\}}(\Omega)}); \quad (3.25)$$

if ϕ_n *is the solution of* Algorithm 3.1,

$$\|\phi - \phi_n\|_{H^1(\Omega)} \leq Ch^{\min\{k,\beta+2,\alpha+1,2\alpha\}}(D_1 + \|\phi\|_{H^{\min\{k+1,\beta+3,\alpha+2\}}(\Omega)}). \quad (3.26)$$

The proof is given in Appendix A.2. We also have the following L^2 error estimates.

Theorem 3.9 *Let* ϕ_n *be the finite element solution of* (3.6) *from* Algorithm 3.2 *or* Algorithm 3.1, *and* ϕ *be the solution of the biharmonic problem* (1.1). *If* ϕ_n *is the solution of* Algorithm 3.2, *then we have*

$$\|\phi - \phi_n\| \leq Ch^{\min\{k+1,\alpha+2,2\alpha\}}(D_2 + \|\phi\|_{H^{\min\{k+1,\alpha+2\}}(\Omega)}); \quad (3.27)$$

if ϕ_n *is the solution of* Algorithm 3.1, *then we have*

$$\|\phi - \phi_n\| \leq Ch^{\min\{k+1,\beta+3,\alpha+2,2\alpha\}}(D_1 + \|\phi\|_{H^{\min\{k+1,\beta+3,\alpha+2\}}(\Omega)}). \quad (3.28)$$

The proof is given in Appendix A.3.

Remark 3.10 For results in Theorems 3.8 and 3.9, we have the following comparisons.

- Recall that ω is the largest interior angle of domain Ω. From Fig. 2(b), we find $\alpha < \beta + 1$ when $\omega > \tilde{\omega} \approx 0.35481\pi$, which together with Theorems 3.8 and 3.9 implies that when $\omega > \tilde{\omega}$, Algorithms 3.1 and 3.2 give the same H^1 and L^2 convergence rates. In particular, these two algorithms always give the same convergence rates on non-convex domains.
- The inequality $\alpha \geq \beta + 1$ happens only when $\omega \leq \tilde{\omega}$. However, we found by Fig. 2(a) when $\omega \leq \tilde{\omega}$, it follows $\beta + 1 > 4$, which implies that Algorithms 3.1 and 3.2 also give the same convergence rates for $k \leq 4$.

As a comparison to the existing error estimates shown in (3.7), we summarize a special case discussed in this subsection.

Corollary 3.11 *If the largest interior angle of the domain* $\omega \in (\pi, 2\pi)$, *it follows* $w \in H^{1+\beta}(\Omega)$, $\mathbf{u} \in [H^{1+\alpha}(\Omega)]^2$, $p \in H^\alpha(\Omega)$, *and* $\phi \in H^{2+\alpha}(\Omega)$, $0 < \alpha < \beta < 1$, *and the finite approximations in both* Algorithms 3.2 and 3.1 *satisfy*

$$\begin{aligned} h^\beta\|w - w_n\|_{H^1(\Omega)} + \|w - w_n\| &\leq Ch^{2\beta}, \\ \|\mathbf{u} - \mathbf{u}_n\|_{[H^1(\Omega)]^2} + \|p - p_n\| &\leq Ch^\alpha, \\ \|\mathbf{u} - \mathbf{u}_n\|_{[L^2(\Omega)]^2} + \|\phi - \phi_n\|_{H^1(\Omega)} + \|\phi - \phi_n\| &\leq Ch^{2\alpha}. \end{aligned} \quad (3.29)$$

Remark 3.12 Assume that $u \in H^{2+\alpha}(\Omega)$. The standard error estimates (3.7) gives [21, 28]

$$\|\phi - \phi_n\|_{H^1(\Omega)} \leq Ch^\alpha. \quad (3.30)$$

However, thanks to Lemma 3.6, we can obtain a more refined estimate for the same ϕ_n by Theorem 3.8 and Corollary 3.11,

$$\|\phi - \phi_n\|_{H^1(\Omega)} \leq Ch^{2\alpha}. \tag{3.31}$$

In addition, the convergence rates in Corollary 3.11 are verified by the numerical results presented in Example 5.2.

4 Optimal error estimates on graded meshes

To improve the convergence rate, we consider Algorithm 3.2 on graded meshes, beginning with a review of key concepts.

4.1 Weighted Sobolev space and graded meshes

From now on, we denote the vertices of domain Ω by Q_i, $i = 1, \cdots, N$. Let $r_i = r_i(x, Q_i)$ be the distance from x to Q_i. Define

$$\rho(x) = \Pi_{1\leq i\leq N} r_i(x, Q_i). \tag{4.1}$$

Let $\mathbf{a} = (a_1, \cdots, a_i, \cdots, a_N)$ be a vector where each component a_i associated with Q_i. For a scalar t, we denote $t + \mathbf{a} = (t + a_1, \cdots, t + a_N)$. Using this notation, we define

$$\rho(x)^{(t+\mathbf{a})} = \Pi_{1\leq i\leq N} r_i^{(t+\mathbf{a})}(x, Q_i) = \Pi_{1\leq i\leq N} r_i^t(x, Q_i)\Pi_{1\leq i\leq N} r_i^{a_i}(x, Q_i) = \rho(x)^t \rho(x)^{\mathbf{a}}.$$

We then introduce the Kondratiev-type weighted Sobolev spaces to facilitate the analysis of the Stokes problem (2.12) and the Poisson problem (2.40).

Definition 4.1 [44] For $a \in \mathbb{R}$, $m \geq 0$, and $G \subset \Omega$, we define the Kondratiev-type weighted Sobolev space as

$$\mathcal{K}_{\mathbf{a}}^m(G) := \{v|\ \rho^{|\nu|-\mathbf{a}}\partial^\nu v \in L^2(G), \forall\ |\nu| \leq m\},$$

where $\nu = (\nu_1, \nu_2) \in \mathbb{Z}_{\geq 0}^2$ is a multi-index, $|\nu| = \nu_1 + \nu_2$, and $\partial^\nu = \partial_x^{\nu_1}\partial_y^{\nu_2}$. The norm associated with $\mathcal{K}_{\mathbf{a}}^m(G)$ is given by

$$\|v\|_{\mathcal{K}_{\mathbf{a}}^m(G)} = \left(\sum_{|\nu|\leq m} \iint_G |\rho^{|\nu|-\mathbf{a}}\partial^\alpha v|^2 dxdy\right)^{\frac{1}{2}}.$$

To enhance the convergence rate of the numerical approximation from Algorithms 3.1 and 3.2, we introduce the algorithm of graded meshes.

Fig. 3 Refinement of an edge AB. Left (midpoint refinement): $A \neq Q_i$ and $B \neq Q_i$ ($|AD| = |BD|$). Right (graded refinement): $A = Q_i$ ($|AD| = \kappa_{Q_i}|AB|$, $\kappa_{Q_i} \in (0, 0.5]$)

Algorithm 4.2 (Graded refinements) Given a shape-regular triangulation $\mathcal{T}$ of Ω, let AB be an edge in $\mathcal{T}$ with endpoints A and B. A graded refinement introduces a new node D on AB according to the following rules (see Fig. 3):

1. (Neither A nor B coincides with any Q_i): The node D is placed at the midpoint of AB, i.e., $|AD| = |BD|$.
2. (A coincides with a vertex Q_i): The node D is placed such that $|AD| = \kappa_{Q_i}|AB|$, where $\kappa_{Q_i} \in (0, 0.5]$ is a parameter.

For each triangle $T = \triangle x_0x_1x_2 \in \mathcal{T}$, a new node is placed on each edge based on these two rules. The triangle T is then refined into four smaller triangles by connecting these newly introduced nodes, as illustrated in Fig. 4. The collection of all these smaller triangles forms the graded refinement, $\kappa(\mathcal{T})$.

Given a grading parameter κ_{Q_i}, Algorithm 4.2 can produce smaller elements near Q_i for a better approximation of the singular solution. It is an explicit construction of graded meshes based on recursive refinements. See also [2, 7, 39, 40, 42] and references therein for more discussions on graded meshes. Starting from an initial mesh $\mathcal{T}_0$ that meets these criteria, the sequence of graded meshes $\mathcal{T}_n$, $n \geq 0$ is generated recursively

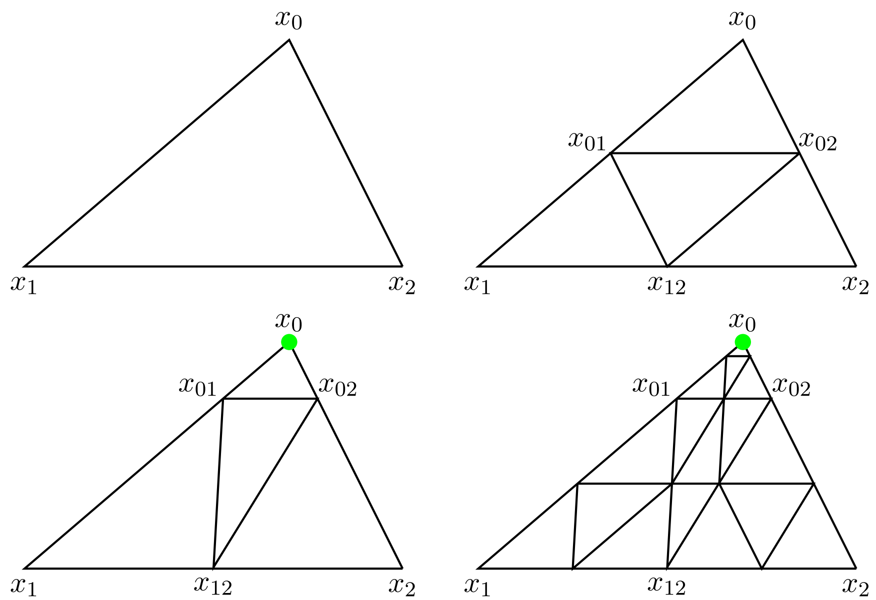


Fig. 4 Midpoint refinement and graded refinements of a triangle $T = \triangle x_0x_1x_2$. Top left: the initial triangle. Top right: the midpoint refinement. Bottom left: the graded refinement toward $x_0 = Q_i$ with $\kappa_{Q_i} \in (0, 0.5]$. Bottom: two consecutive graded refinements

through

$$\mathcal{T}_{n+1} = \kappa(\mathcal{T}_n).$$

Note that after n refinements, the number of triangles in the mesh $\mathcal{T}_n$ is $O(4^n)$, so we denote the "mesh size" of $\mathcal{T}_n$ by

$$h = 2^{-n}. \tag{4.2}$$

In Algorithm 4.2, we choose the parameter κ_{Q_i} for each vertex Q_i as follows. To better observe the threshold of grading parameter κ_{Q_i} in obtaining the optimal convergence rates, we always assume that the degree of polynomials k in Algorithm 3.2 satisfies $k \le m$ in the following discussions. Otherwise, we replace k by $\min\{k, m\}$.

Let α_0^i, $i = 1, \cdots, N$ be the solution of (2.3) with ω being replaced by the interior angle ω_i at Q_i. It is obvious that $\alpha_0 = \min_i \alpha_0^i$. For the rest of this paper, we choose

$$\kappa_{Q_i} = 2^{-\frac{\theta}{a_i}} \left(\le \frac{1}{2} \right), \tag{4.3}$$

where $0 < a_i < \alpha_0^i$ and $a_i \le \theta$, $i = 1, \cdots, N$, and $\theta \in (0, k]$ is a parameter to be specified later (see, e.g., Remark 4.10, Theorems 4.11, and 4.12). Note that close to the vertex Q_i if $a_i = \theta$, the grading parameter $\kappa_{Q_i} = \frac{1}{2}$, or equivalently, the mesh is a quasi-uniform mesh.

In the following, we generalize the projection errors on graded meshes in [42, 43].

Lemma 4.3 *(Projection errors on graded meshes) Let $\mathcal{T}_0$ be an initial triangle of the triangulation $\mathcal{T}_n$ in* Algorithm 4.2 *with grading parameters κ_{Q_i} in* (4.3). *For $1 \le k \le m$, if $v_I \in V_n^k$ (resp. $q_I \in V_n^{k-1}$) be the nodal interpolation of $v \in \mathcal{K}_{\mathbf{a}+1}^{m+1}(\Omega)$ (resp. $q \in \mathcal{K}_{\mathbf{a}}^{m}(\Omega)$). Then, it follows the following interpolation error*

$$\|v - v_I\|_{H^1(\Omega)} \le Ch^\theta \|v\|_{\mathcal{K}_{\mathbf{a}+1}^{m+1}(\Omega)}, \quad \|q - q_I\| \le Ch^\theta \|q\|_{\mathcal{K}_{\mathbf{a}}^{m}(\Omega)}, \tag{4.4}$$

where $h := 2^{-n}$, and θ is given in (4.3).

4.2 Regularity in weighted Sobolev space

In Section 2, we have introduced the decomposition of the biharmonic problem (1.1) into the Poisson and Stokes problems. Before presenting the regularity for the decomposed problems in the weighted Sobolev space, we first recall the following regularity results.

Let $\mathbf{a}, \mathbf{b}$ be vectors of the same dimension N, then we say $\mathbf{a} < \mathbf{b}$ if $a_i < b_i$, $i = 1, \cdots, N$, where a_i, b_i are the ith entry of $\mathbf{a}, \mathbf{b}$, respectively. We also denote the vector $\boldsymbol{\alpha}_0 = (\alpha_0^1, \cdots, \alpha_0^i, \cdots, \alpha_0^N)$, where α_0^i given in the section above. For the Stokes problem (2.10), we have the following regularity estimate in weighted Sobolev space [9, 35].

Lemma 4.4 *Let* $(\mathbf{u}, p) \in [H_0^1(\Omega)]^2 \times L_0^2(\Omega)$ *be the solution of the Stokes problem* (2.10)*. For* $m \geq 1$ *and* $0 < \mathbf{a} < \boldsymbol{\alpha}_0$*, if* $\mathbf{F} \in [\mathcal{K}_{\mathbf{a}-1}^{m-1}(\Omega)]^2$*, then it follows*

$$\|\mathbf{u}\|_{[\mathcal{K}_{\mathbf{a}+1}^{m+1}(\Omega)]^2} + \|p\|_{\mathcal{K}_{\mathbf{a}}^{m}(\Omega)} \leq C\|\mathbf{F}\|_{[\mathcal{K}_{\mathbf{a}-1}^{m-1}(\Omega)]^2}. \tag{4.5}$$

Assume that no ω_i satisfies neither $\omega_i = \tan\omega_i$ nor (2.4). Then we have the following result.

Lemma 4.5 *Assume that* $f \in \mathcal{K}_{\mathbf{a}-2}^{m-2}(\Omega) \cap H^{-1}(\Omega)$ *for* $m \geq 1$ *and let* $\phi \in H_0^2(\Omega)$ *be the solution of biharmonic problem* (1.1) *or the solution of the Possible problem* (2.40)*. Then it follows* $\phi \in \mathcal{K}_{\mathbf{a}+2}^{m+2}(\Omega)$.

Proof By Lemma 2.8 and Lemma 2.9, the solution of the Poisson problem (2.40) shares the same solution as the biharmonic problem (1.1). Therefore, the conclusion follows from [35, Theorem 7.4.3]. □

4.3 Optimal error estimates on graded meshes

In this section, we assume that $f \in \mathcal{K}_{\mathbf{a}-1}^{m-1}(\Omega) \cap \mathcal{K}_{\mathbf{b}-1}^{m-1}(\Omega) \subset \mathcal{K}_{\mathbf{a}-2}^{m-2}(\Omega)$ with $0 < \mathbf{a} < \boldsymbol{\alpha}_0$, and $0 < \mathbf{b} < \boldsymbol{\beta}_0$, where $\boldsymbol{\beta}_0 = (\frac{\pi}{\omega_1}, \cdots, \frac{\pi}{\omega_N})$.

The regularity estimate [7] for the Poisson problem (2.16) on weighted Sobolev space gives

$$\|w\|_{\mathcal{K}_{\mathbf{b}+1}^{m+1}(\Omega)} \leq C\|f\|_{\mathcal{K}_{\mathbf{b}-1}^{m-1}(\Omega)}. \tag{4.6}$$

Since the bilinear form in (3.3) is coercive and continuous on V_n^k, we have by Céa's Theorem,

$$\|w - w_n\|_{H^1(\Omega)} \leq C \inf_{v \in V_n^k} \|w - v\|_{H^1(\Omega)}. \tag{4.7}$$

Recall that $\alpha_0 = \min_i\{\alpha_0^i\}$ given by (2.3), and $\beta_0 = \min_i\{\beta_0^i\} = \frac{\pi}{\omega}$ are the threshold values corresponding to the largest interior angle ω, then we have the following result.

Lemma 4.6 *Assume that the grading parameters* κ_{Q_i} *are given by* (4.3)*. Let* $w_n \in V_n^k$ *be the finite element solution of* (3.3)*, and* w *is the solution of the Poisson problem* (2.16)*, then it follows*

$$\|w - w_n\|_{H^1(\Omega)} \leq Ch^{\theta'}\|w\|_{\mathcal{K}_{\mathbf{b}+1}^{\theta'+1}(\Omega)}, \quad \|w - w_n\| \leq Ch^{\min\{2\theta', \theta'+1\}}\|w\|_{\mathcal{K}_{\mathbf{b}+1}^{\theta'+1}(\Omega)} \tag{4.8}$$

where $h := 2^{-n}$ *and* $\theta' = \min\left\{\max\left\{\frac{\beta_0}{\alpha_0}\theta, \beta\right\}, k\right\}$ *with* β *given in* (2.9).

The proof is given in (8.1). Thus, we have the following result,

$$
\begin{aligned}
\|\mathbf{F}-\mathbf{F}_n\|_{[L^2(\Omega)]^2} =&\|\mathbf{curl}\, v-\mathbf{curl}\, v_n\|_{[L^2(\Omega)]^2} \le \|w-w_n\|_{H^1(\Omega)} \le Ch^{\theta'}\|w\|_{\mathcal{K}^{\theta'+1}_{\mathbf{b}+1}(\Omega)},\\
\|\mathbf{F}-\mathbf{F}_n\|_{[H^{-1}(\Omega)]^2} =&\|\mathbf{curl}\, w-\mathbf{curl}\, w_n\|_{[H^{-1}(\Omega)]^2} \le C\|w-w_n\| \le Ch^{\min\{\theta'+1,2\theta'\}}\|w\|_{\mathcal{K}^{\theta'+1}_{\mathbf{b}+1}(\Omega)}.
\end{aligned}
\tag{4.9}
$$

Now we have the following error estimate of the Mini element approximation ($k=1$) or Taylor-Hood method ($k\ge 2$) in Algorithm 3.2 on graded meshes for the Stokes problem (2.10).

Lemma 4.7 *The bilinear forms in both the Mini element method and the Taylor-Hood method on graded meshes satisfy the LBB or inf-sup condition.*

Proof For given $\kappa=\min_i\{\kappa_{Q_i}\}$, Algorithm 4.2 implies that there exists a constant $\sigma(\kappa)>0$ such that

$$h_T \le \sigma(\kappa)\rho_T, \quad \forall T\in\mathcal{T}_n, \tag{4.10}$$

where h_T is the diameter of T, and ρ_T is the maximum diameter of all circles contained in T. Under condition (4.10) of the graded mesh, the conclusion follows from [51, Theorem 3.1]. □

In this section, to simplify the notation, we define

$$E_2 =\|\mathbf{u}\|_{[\mathcal{K}^{\theta+1}_{\mathbf{a}+1}(\Omega)]^2}+\|p\|_{\mathcal{K}^{\theta}_{\mathbf{a}}(\Omega)}, \tag{4.11}$$

$$E_1 =\|\mathbf{u}\|_{[\mathcal{K}^{\min\{\theta,\theta'+1\}+1}_{\mathbf{a}+1}(\Omega)]^2}+\|p\|_{\mathcal{K}^{\min\{\theta,\theta'+1\}}_{\mathbf{a}}(\Omega)}+\|w\|_{\mathcal{K}^{\theta'+1}_{\mathbf{b}+1}(\Omega)}. \tag{4.12}$$

Then the following result follows.

Theorem 4.8 *Assume that the grading parameters κ_{Q_i} are given by (4.3) and $\theta\in(0,k]$. Let $(\mathbf{u},p)$ be the solution of the Stokes problem (2.12), and $(\mathbf{u}_n,p_n)$ be the Mini element solution ($k=1$) or Taylor-Hood element solution ($k\ge 2$) on graded meshes $\mathcal{T}_n$. If $(\mathbf{u}_n,p_n)$ is the solution in* Algorithm 3.2, *then it follows that*

$$\|\mathbf{u}-\mathbf{u}_n\|_{[H^1(\Omega)]^2}+\|p-p_n\|\le Ch^{\theta}E_2; \tag{4.13}$$

if it is the solution of in Algorithm 3.1, *then it follows*

$$\|\mathbf{u}-\mathbf{u}_n\|_{[H^1(\Omega)]^2}+\|p-p_n\|\le Ch^{\min\{\theta,\theta'+1\}}E_1, \tag{4.14}$$

where $h:=2^{-n}$ and θ' is given in Lemma 4.6.

In weighted Sobolev space, the regularity result for (3.20) with $l=0,1$ has the form

$$\|\mathbf{r}\|_{[\mathcal{K}^{l+2}_{\mathbf{a}'+1}(\Omega)]^2}+\|s\|_{\mathcal{K}^{l+1}_{\mathbf{a}'}(\Omega)}\le C\|\mathbf{g}\|_{[\mathcal{K}^{l}_{\mathbf{a}'-1}(\Omega)]^2}\le C\|\mathbf{g}\|_{[\mathcal{K}^{l}_{\mathbf{b}}(\Omega)]^2}, \tag{4.15}$$

where $0 < \mathbf{a}' = \min\{\mathbf{a}, \mathbf{b}+1\}$ with $0 < \mathbf{a} < \boldsymbol{\alpha}_0$ and $0 < \mathbf{b} < \boldsymbol{\beta}_0$.

Now our estimates for the Stokes equation are as follows.

Theorem 4.9 *Assume that the grading parameters κ_{Q_i} are given by (4.3) and $\theta \in (0, k]$. Let $(\mathbf{u}, p)$ be the solution of the Stokes problem (2.12), and $(\mathbf{u}_n, p_n)$ be the Mini element solution ($k = 1$) or Taylor-Hood element solution ($k \geq 2$) in* Algorithm 3.1 *on graded meshes $\mathcal{T}_n$. If $(\mathbf{u}_n, p_n)$ is the solution in* Algorithm 3.2*, then it follows*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_n\|_{[L^2(\Omega)]^2} &\leq C h^{\min\{2\theta, \theta+1\}} E_2, \\ \|\mathbf{u} - \mathbf{u}_n\|_{[(\mathcal{K}^1_{\mathbf{b}}(\Omega))^*]^2} &\leq C h^{\min\{2\theta, \theta+2\}} E_2; \end{aligned} \tag{4.16}$$

if it is the solution in Algorithm 3.2*, then it follows*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_n\|_{[L^2(\Omega)]^2} &\leq C h^{\min\{2\theta, \theta+1, \theta'+2\}} E_1, \\ \|\mathbf{u} - \mathbf{u}_n\|_{[(\mathcal{K}^1_{\mathbf{b}}(\Omega))^*]^2} &\leq C h^{\min\{2\theta, \theta+2, k+\theta', \theta+\theta'+1, \theta'+3\}} E_1, \end{aligned} \tag{4.17}$$

where $h := 2^{-n}$ and θ' is given in Lemma 4.6*. Here, $(\cdot)^*$ represents the dual space.*

The proof is given in Appendix B.3.

Remark 4.10 By Theorems 4.8 and 4.9, we find that if we take

$$\theta = k \tag{4.18}$$

in the grading parameter κ_{Q_i}, then we can obtain the optimal convergence rate for the Stokes approximations in Algorithm 3.2,

$$\|\mathbf{u} - \mathbf{u}_n\|_{[H^1(\Omega)]^2} + \|p - p_n\| \leq C h^k E_2, \tag{4.19a}$$

$$\|\mathbf{u} - \mathbf{u}_n\|_{[L^2(\Omega)]^2} \leq C h^{k+1} E_2. \tag{4.19b}$$

For Algorithm 3.1, it follows

$$\|\mathbf{u} - \mathbf{u}_n\|_{[H^1(\Omega)]^2} + \|p - p_n\| \leq C h^{\min\left\{\max\left\{\frac{\beta_0}{\alpha_0}\theta, \beta\right\}+1, k\right\}} E_1, \tag{4.20a}$$

$$\|\mathbf{u} - \mathbf{u}_n\|_{[L^2(\Omega)]^2} \leq C h^{\min\left\{k+1, \max\left\{\frac{\beta_0}{\alpha_0}\theta, \beta\right\}+2\right\}} E_1. \tag{4.20b}$$

However, to obtain the optimal convergence rate for the biharmonic approximation, the convergence rates of the Mini element or the Taylor-Hood element approximations don't have to be optimal. Therefore, we shall figure out the admissible parameters θ such that the convergence rate of the biharmonic approximation is optimal.

Recall α, β given in (2.9). Now we have the main results for the biharmonic equation.

Theorem 4.11 *Assume that the grading parameters κ_{Q_i} are given by (4.3) with*

$$\theta = \max\left\{k - 1, \min\{k, \alpha\}\right\}. \tag{4.21}$$

Let $\phi_n \in V_n^k$ be the finite element solution of (3.6)*, and ϕ be the solution of the biharmonic problem* (1.1)*. If ϕ_n is the solution in* Algorithm 3.2*, then it follows*

$$\|\phi - \phi_n\|_{H^1(\Omega)} \leq Ch^k(E_2 + \|\phi\|_{\mathcal{K}^{k+2}_{\mathbf{a}+2}(\Omega)}); \tag{4.22}$$

if ϕ_n is the solution in Algorithm 3.1*, then it follows*

$$\|\phi - \phi_n\|_{H^1(\Omega)} \leq Ch^k(E_1 + \|\phi\|_{\mathcal{K}^{k+2}_{\mathbf{a}+2}(\Omega)}) \quad \text{if (i) } k \leq 4; \text{ or (ii) } k > 4 \text{ and } \omega > \pi; \tag{4.23a}$$

$$\|\phi - \phi_n\|_{H^1(\Omega)} \leq Ch^{\min\left\{k, \max\left\{\frac{\beta_0}{\alpha_0}(k-1)+2, \beta+2\right\}\right\}}(E_1 + \|\phi\|_{\mathcal{K}^{k+2}_{\mathbf{a}+2}(\Omega)}) \quad \text{if } k > 4 \text{ and } \frac{\pi}{3} \leq \omega < \pi, \tag{4.23b}$$

where $h := 2^{-n}$ and $\omega \in [\pi/3, 2\pi)\backslash\{\pi\}$ is the largest interior angle of domain Ω.

The proof is given in Appendix B.4.

Theorem 4.12 *Assume that the grading parameters κ_{Q_i} are given by* (4.3) *with*

$$\theta = \max\left\{k-1, \frac{k+1}{2}, \min\{k, \alpha\}\right\}. \tag{4.24}$$

Let $\phi_n \in V_n^k$ be the finite element solution of (3.6)*, and ϕ be the solution of the biharmonic problem* (1.1)*. If ϕ_n is the solution in* Algorithm 3.2*, then it follows*

$$\|\phi - \phi_n\| \leq Ch^{k+1}(E_2 + \|\phi\|_{\mathcal{K}^{k+2}_{\mathbf{a}+2}(\Omega)}); \tag{4.25}$$

if ϕ_n is the solution in Algorithm 3.1*, then it follows*

$$\|\phi - \phi_n\| \leq Ch^{k+1}(E_1 + \|\phi\|_{\mathcal{K}^{k+2}_{\mathbf{a}+2}(\Omega)}) \quad \text{if (i) } k \leq 4; \text{ or (ii) } k > 4 \text{ and } \omega > \pi; \tag{4.26a}$$

$$\|\phi - \phi_n\| \leq Ch^{\min\left\{k+1, \max\left\{\frac{\beta_0}{\alpha_0}(k-1)+2, \beta+2\right\}+1\right\}}(E_1 + \|\phi\|_{\mathcal{K}^{k+2}_{\mathbf{a}+2}(\Omega)}) \quad \text{if } k > 4 \text{ and } \frac{\pi}{3} \leq \omega < \pi, \tag{4.26b}$$

where $h := 2^{-n}$ and $\omega \in [\pi/3, 2\pi)\backslash\{\pi\}$ is the largest interior angle of domain Ω.

The proof is given in Appendix B.5.

Remark 4.13 For the results in Theorems 4.11 and 4.12, we have the following remarks.

- If $k = 1$ in (4.21), then $a_i = \theta, i = 1, \cdots, N$ gives $\kappa_{Q_i} = \frac{1}{2}$, which indicates the mesh is exactly the quasi-uniform mesh.
- For $\frac{\pi}{3} \leq \omega < \pi$, if θ in the grading parameter κ_{Q_i} is modified as

$$\theta = \max\left\{k-1, \min\{\alpha, k\}, \frac{\beta_0}{\alpha_0}\frac{(k-1)^2}{k-2}, \frac{\beta_0}{\alpha_0}\frac{(k-1)}{k-2}\min\{\alpha, k\}\right\}, \quad \forall\, k > 4,$$

then the estimates in (4.23b) and (4.26b) will separately become

$$\begin{aligned} \|\phi - \phi_n\|_{H^1(\Omega)} \le & Ch^k, \\ \|\phi - \phi_n\| \le & Ch^{k+1}. \end{aligned}$$

5 Numerical illustrations

This section examines the theoretical findings. When the exact solution (or vector) v is unknown, we estimate the numerical convergence rate using

$$\mathcal{R} = \log_2 \frac{|v_j - v_{j-1}|_{[H^l(\Omega)]^{l'}}}{|v_{j+1} - v_j|_{[H^l(\Omega)]^{l'}}}, \tag{5.1}$$

serving as an indicator of the actual convergence rate [41]. Here, v_j denotes the finite element solution on the mesh $\mathcal{T}_j$, which is obtained after j refinements of the initial triangulation $\mathcal{T}_0$. For scalar functions, we take $l' = 1$, otherwise, $l' = 2$. So if $v_j = w_j, \phi_j, p_j$, we take $l' = 1$; if $v_j = \mathbf{u}_j$, we take $l' = 2$.

To test the performance of Algorithms 3.1 and 3.2 for solving the biharmonic problem (1.1), we shall use the H^2-conforming Argyris finite element approximation [3] with the boundary conditions being weakly enforced by the Nitsche's method as a reference solution ϕ_R, which is computed on the same mesh as Algorithms 3.1 and 3.2. Since the solution of the H^2-conforming finite element method converges to the exact solution ϕ regardless of the convexity of the domain as the mesh is refined, we can use ϕ_R as a good approximation of the exact solution ϕ.

Since the convergence rate of the finite element approximation w_j of the Poisson equation in Algorithm 3.1 has been well investigated in many papers (see e.g., [43, 45]), we will not report the convergence rates of w_j in the numerical tests.

Example 5.1 We solve the biharmonic problem (1.1) with $f = 1$ using Algorithm 3.2 based on polynomials with $k = 2$ on quasi-uniform meshes obtained by the midpoint refinements. The source term of the involved Stokes problem (2.10) in Algorithm 3.2 is taken as $\mathbf{F} = (0, x)^T$, which satisfies (2.11).

Test case 1. We first consider this problem in a square domain $\Omega = (-1, 1)^2$ with the initial mesh given in Fig. 5(a). The errors in L^∞ norm between the finite element solution ϕ_j and the reference solution ϕ_R are given in Table 2. The finite element solution and its difference from the reference solution are shown in Fig. 5(b) and (c),

Table 2 $\|\phi_R - \phi_j\|_{L^\infty(\Omega)}$ in the square domain on quasi-uniform meshes

	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$
$k=2$	6.66250e-05	9.10404e-06	1.17798e-06	1.49996e-07	1.89341e-08	2.37885e-09

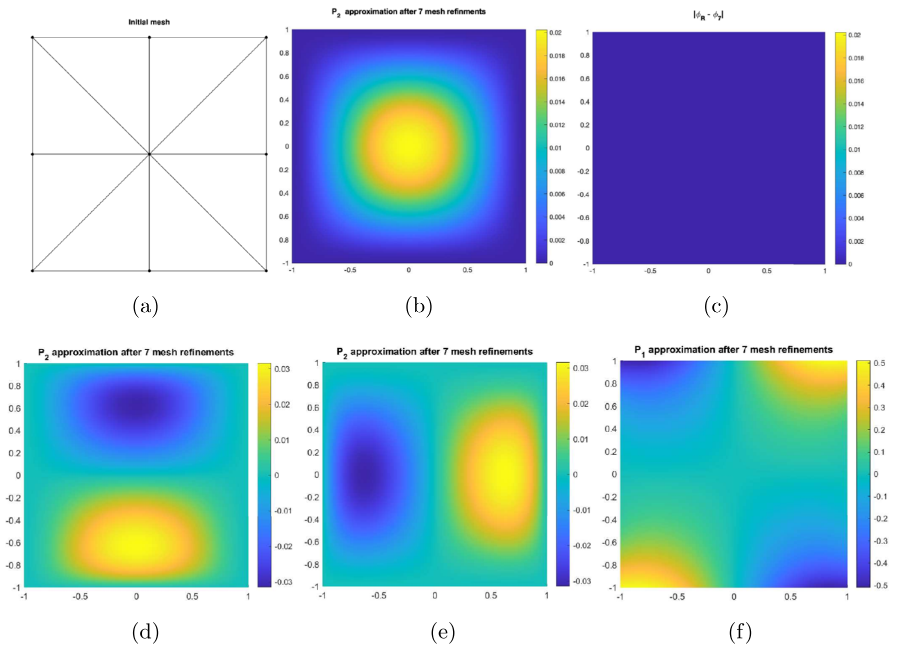


Fig. 5 Example 5.1 Test case 1: (a) the initial mesh; (b) P_2 finite element approximation ϕ_7; (c) the difference $|\phi_R - \phi_7|$; (d) Taylor-Hood element approximation u_1 of $\mathbf{u}_7$; (e) Taylor-Hood element approximation u_2 of $\mathbf{u}_7$; (f) Taylor-Hood element approximation p_7

respectively. These results indicate that the solution of Algorithm 3.2 converges to the exact solution. In Table 3, we report the H^1 and L^2 convergence rates of the finite element solutions ϕ_j, respectively. The results indicate that optimal convergence rates are obtained for finite element solutions of the biharmonic problem.

In addition, the $P_2 - P_1$ Taylor-Hood approximations $\mathbf{u}_7$ and p_7 for the involved Stokes problem are shown in Fig. 5(d)-(f). In Table 3, we also report the H^1 and/or L^2 convergence rates of the Taylor-Hood element approximations $\mathbf{u}_j$ and p_j. The results imply that optimal convergence rates are obtained for Taylor-Hood element approximations of the Stokes problem.

Table 3 Numerical convergence rates in the square domain on quasi-uniform meshes

		H^1 rate of ϕ_j	L^2 rate of ϕ_j	H^1 rate of $\mathbf{u}_j$	L^2 rate of $\mathbf{u}_j$	L^2 rate of p_j
$k = 2$	$j = 4$	1.96	3.00	1.99	3.01	2.03
	$j = 5$	1.99	3.00	2.00	3.02	2.02
	$j = 6$	2.00	3.00	2.00	3.01	2.01
	$j = 7$	2.00	3.00	2.00	3.00	2.00
	$j = 8$	2.00	3.00	2.00	3.00	2.00

Table 4 Numerical convergence rates in the square domain on quasi-uniform meshes

		L^∞ rate of ϕ_j	L^∞ rate of $\mathbf{u}_j$
$k=2$	$j=7$	2.98	2.75
	$j=8$	2.99	2.81
	$j=9$	2.99	2.89
	$j=10$	3.00	2.93

These results are consistent with our expectation in Lemma 3.4, Theorems 3.8, and 3.9 for the biharmonic problem (1.1) and the involved Stokes problem (2.10) in a convex domain. In addition, we report the $L^\infty(\Omega)$ convergence rates for ϕ_j and $\mathbf{u}_j$ in Table 4, where the numerical results demonstrate that optimal convergence rates are achieved.

Test case 2. We then consider this problem in an L-shaped domain $\Omega = \Omega_0 \setminus \Omega_1$ with $\Omega_0 = (-1, 1)^2$ and $\Omega_1 = [0, 1) \times (-1, 0]$ based on the initial mesh given in Fig. 6(a). The error $\|\phi_R - \phi_j\|_{L^\infty(\Omega)}$ is given in Table 5. The finite element solution and its difference from the reference solution are shown in Fig. 6(b) and (c), respectively. These results indicate that the solutions of Algorithm 3.2 converge to the exact solution in a nonconvex polygonal domain.

Example 5.2 We solve the problem in Example 5.1 Test Case 2 using both Algorithm 3.1 and Algorithm 3.2 with polynomials $k = 1, 2$ on a sequence of graded meshes (including quasi-uniform mesh). We denote the graded parameter κ_{Q_i} in (4.3) associated with the reentrant corner by $\kappa = \frac{\theta}{a}$ for $a < \alpha_0$. The initial mesh and the graded mesh after 2 mesh refinements are shown in Figs. 6(a) and 7(a), respectively.

In Table 6 (Algorithm 3.2) and Table 7 (Algorithm 3.1), we show the numerical convergence rates of the finite element approximation ϕ_j in both H^1 and L^2 norms to the solution of the biharmonic problem. We note that the convergence rates from both Algorithms 3.2 and 3.1 are almost the same. From Tables 6 and 7, we find on quasi-uniform meshes ($\kappa = 0.5$) that the H^1 convergence rate of the P_1 finite element approximation is optimal with $\mathcal{R} = 1$, and that of the P_2 finite element approximations is suboptimal with $\mathcal{R} \approx 1.10$. Both of them are consistent with the theoretical result in Theorem 3.8 in an L-shaped domain, that is $\mathcal{R}_{\text{exact}} = \min\{k, \alpha_0 + 1, 2\alpha_0\} \approx \min\{k, 1.09\}$ for Algorithm 3.2, and $\mathcal{R}_{\text{exact}} = \min\{k, \beta_0+2, \alpha_0+1, 2\alpha_0\} \approx \min\{k, 1.09\}$ for Algorithm 3.1, where α_0 is given in Table 1 with $\omega = \frac{3\pi}{2}$, and $\beta_0 = \frac{2}{3}$. We also find that the convergence rates of P_1 finite element approximations are optimal with $\mathcal{R} = 1$ on graded meshes with $\kappa < 0.5$, and that of the P_2 finite

Table 5 The L^∞ error $\|\phi_R - \phi_j\|_{L^\infty(\Omega)}$ in the L-shaped domain on quasi-uniform meshes

	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$
$k=2$	8.74987e-04	3.94122e-04	1.77980e-04	8.26205e-05	3.86434e-05	1.81330e-05

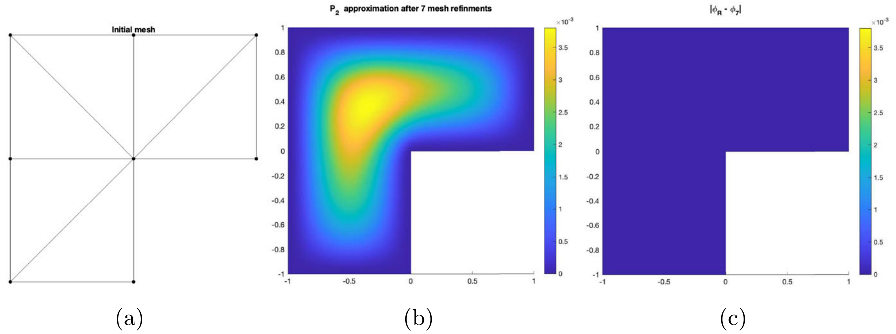


Fig. 6 Example 5.1 Test case 2: (a) the initial mesh; (b) P_2 finite element approximation ϕ_7; (c) the difference $|\phi_R - \phi_7|$

element approximations are optimal with $\mathcal{R} = 2$ on graded meshes with $\kappa < 0.3$, which are consistent with the theoretical result in Theorem 4.11. Namely, the optimal convergence rate can be obtained when $\kappa = 2^{-\frac{\alpha}{a}} \le 2^{-\frac{\alpha}{\alpha}} = 0.5$ for P_1 finite element approximations, and $\kappa = 2^{-\frac{1}{a}} < 2^{-\frac{1}{\alpha_0}} \approx 0.28$ for P_2 finite element approximations.

Again from Tables 6 and 7, the L^2 convergence rates of both P_1 and P_2 finite element approximations on quasi-uniform meshes are suboptimal with $\mathcal{R} \approx 1.14$ and $\mathcal{R} \approx 1.09$, which are consistent with the theoretical result $\mathcal{R}_{\text{exact}} \approx \min\{k+1, 1.09\}$ for both Algorithms 3.2 and 3.1 in Theorem 3.9. On graded meshes, the convergence rates of P_1 finite element approximations are optimal with $\kappa \le 0.2$, and those of the P_2 finite element approximations are optimal with $\kappa \le 0.1$, which are consistent with the theoretical result in Theorem 4.12. Namely, the optimal convergence rate $\mathcal{R} = 2$ can be obtained when $\kappa \le 2^{-\frac{1}{\alpha_0}} \approx 0.28$ for P_1 finite element approximations, and $\mathcal{R} = 3$ can be achieved when $\kappa \le 2^{-\frac{1.5}{\alpha_0}} \approx 0.15$ for P_2 finite element approximations.

The Taylor-Hood element approximations $\mathbf{u}_7$ and p_7 based on Algorithm 3.2 on quasi-uniform meshes for the involved Stokes problem are shown in Figs. 7(b)-(d). In Tables 8, 9, 10 and 11, we display numerical convergence rates of the Mini element approximations and the Taylor-Hood approximations from both Algorithm 3.2 and Algorithm 3.1 for the involved Stokes problem. The H^1 convergence rates of $\mathbf{u}_j$ and

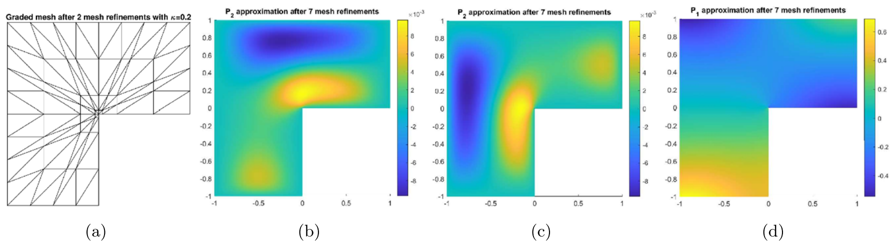


Fig. 7 Example 5.2: (a) the graded mesh after two mesh refinements with $\kappa = 0.2$; (b) Taylor-Hood element approximation u_1 of $\mathbf{u}_7$; (c) Taylor-Hood element approximation u_2 of $\mathbf{u}_7$; (d) Taylor-Hood element approximation p_7

Table 6 Convergence history of finite element approximation ϕ_j of the biharmonic problem from Algorithm 3.2 in the L-shaped domain

	κ	H^1 rate of ϕ_j 0.05	0.1	0.2	0.3	0.4	0.5	L^2 rate of ϕ_j 0.05	0.1	0.2	0.3	0.4	0.5
$k=1$	$j=7$	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	1.96	1.79	1.37
	$j=8$	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	1.96	1.72	1.26
	$j=9$	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	1.95	1.66	1.18
	$j=10$	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	1.95	1.60	1.14
$k=2$	$j=7$	1.99	2.00	2.00	2.00	1.81	1.23	3.03	3.01	2.98	1.90	1.43	1.08
	$j=8$	2.00	2.00	2.00	2.00	1.71	1.15	3.02	3.00	2.96	1.89	1.44	1.08
	$j=9$	2.00	2.00	2.00	2.00	1.61	1.12	3.01	3.00	2.92	1.89	1.44	1.08

the L^2 convergence rates of p_j with $k = 1, 2$ are suboptimal on quasi-uniform meshes with convergence rates $\mathcal{R} \approx 0.54$, which are consistent with the theoretical result $\mathcal{R}_{\text{exact}} = \alpha_0 \approx 0.54$ in Lemma 3.4 and Lemma 3.7 in an L-shaped domain. On graded meshes, the convergence rates are optimal with $\mathcal{R} = 1$ when $\kappa \le 0.2$ for $k = 1$, and $\mathcal{R} = 2$ when $\kappa \le 0.05$ for $k = 2$. These are consistent with the results in Theorem 4.8 and Remark 4.10. Namely, the optimal convergence rate can be achieved when $\kappa \le 2^{-\frac{1}{\alpha_0}} \approx 0.28$ for $k = 1$, and $\kappa \le 2^{-\frac{2}{\alpha_0}} \approx 0.08$ for $k = 2$.

From Tables 8-11, the L^2 convergence rates of the Mini element approximation and the Taylor-Hood approximation $\mathbf{u}_j$ are suboptimal on quasi-uniform meshes with convergence rates $\mathcal{R} \approx 1.13$ and $\mathcal{R} \approx 1.12$, respectively. They are consistent with the theoretical result $\mathcal{R} = 2\alpha_0 \approx 1.09$ in Lemma 3.4 and Lemma 3.7 in an L-shaped domain. On graded meshes, the convergence rates are optimal separately with $\mathcal{R} = 2$ for $\kappa \le 0.2$, and with $\mathcal{R} = 3$ for $\kappa < 0.1$, which are consistent with the theoretical results in Theorem 4.8 and Remark 4.10. Namely, the optimal convergence rate can be achieved when $\kappa \le 2^{-\frac{1}{\alpha_0}} \approx 0.28$ for $k = 1$, and $\kappa \le 2^{-\frac{2}{\alpha_0}} \approx 0.08$ for $k = 2$.

Table 7 Convergence history of finite element approximation ϕ_j of the biharmonic problem from Algorithm 3.1 in the L-shaped domain

	κ	H^1 rate of ϕ_j 0.05	0.1	0.2	0.3	0.4	0.5	L^2 rate of ϕ_j 0.05	0.1	0.2	0.3	0.4	0.5
$k=1$	$j=7$	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	1.96	1.80	1.38
	$j=8$	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	1.96	1.73	1.26
	$j=9$	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	1.96	1.67	1.19
	$j=10$	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	1.95	1.61	1.14
$k=2$	$j=7$	1.99	2.00	2.00	2.00	1.81	1.22	3.03	3.01	2.98	1.90	1.43	1.06
	$j=8$	2.00	2.00	2.00	2.00	1.71	1.15	3.01	3.00	2.96	1.89	1.43	1.07
	$j=9$	2.00	2.00	2.00	2.00	1.61	1.12	3.01	3.00	2.92	1.89	1.44	1.08

Table 8 Convergence history of the Mini element approximations ($k = 1$) of the Stokes problem from Algorithm 3.2 in the L-shaped domain

	H^1 rate of $\mathbf{u}_j$					L^2 rate of $\mathbf{u}_j$					L^2 rate of p_j				
κ	0.1	0.2	0.3	0.4	0.5	0.1	0.2	0.3	0.4	0.5	0.1	0.2	0.3	0.4	0.5
$j = 7$	1.00	1.00	0.98	0.85	0.64	2.01	2.01	1.96	1.71	1.28	1.10	1.35	1.07	0.74	0.56
$j = 8$	1.00	1.00	0.97	0.82	0.60	2.01	2.00	1.95	1.62	1.20	1.05	1.35	1.01	0.73	0.55
$j = 9$	1.00	1.00	0.97	0.80	0.58	2.01	2.00	1.94	1.55	1.16	1.15	1.34	0.98	0.72	0.55
$j = 10$	1.00	1.00	0.97	0.78	0.56	2.00	2.00	1.94	1.51	1.13	1.29	1.33	0.96	0.72	0.55

Table 9 Convergence history of the $P_2 - P_1$ Taylor-Hood element approximations ($k = 2$) of the Stokes problem from Algorithm 3.2 in the L-shaped domain

	H^1 rate of $\mathbf{u}_j$					L^2 rate of $\mathbf{u}_j$					L^2 rate of p_j				
κ	0.05	0.1	0.2	0.3	0.5	0.05	0.1	0.2	0.3	0.5	0.05	0.1	0.2	0.3	0.5
$j = 7$	1.90	1.84	1.31	0.95	0.54	3.03	3.01	2.99	1.93	1.24	1.91	1.82	1.28	0.95	0.55
$j = 8$	1.95	1.83	1.28	0.95	0.54	3.03	3.01	2.98	1.90	1.19	1.97	1.82	1.27	0.95	0.54
$j = 9$	1.97	1.83	1.27	0.95	0.54	3.01	3.00	2.95	1.89	1.15	1.99	1.82	1.27	0.95	0.54
$j = 10$	1.98	1.82	1.27	0.95	0.54	3.01	3.00	2.91	1.89	1.12	1.99	1.82	1.27	0.95	0.54

Table 10 Convergence history of the Mini element approximations ($k = 1$) of the Stokes problem from Algorithm 3.1 in the L-shaped domain

	H^1 rate of $\mathbf{u}_j$					L^2 rate of $\mathbf{u}_j$					L^2 rate of p_j				
κ	0.1	0.2	0.3	0.4	0.5	0.1	0.2	0.3	0.4	0.5	0.1	0.2	0.3	0.4	0.5
$j = 7$	1.00	1.00	0.98	0.85	0.64	2.01	2.00	1.97	1.73	1.29	1.10	1.35	1.07	0.74	0.54
$j = 8$	1.00	1.00	0.97	0.82	0.60	2.01	2.00	1.96	1.64	1.21	1.04	1.35	1.01	0.73	0.54
$j = 9$	1.00	1.00	0.97	0.80	0.58	2.00	2.00	1.95	1.57	1.16	1.15	1.34	0.98	0.72	0.54
$j = 10$	1.00	1.00	0.97	0.78	0.56	2.00	2.00	1.94	1.52	1.13	1.29	1.35	0.96	0.72	0.54

Table 11 Convergence history of the $P_2 - P_1$ Taylor-Hood element approximations ($k = 2$) of the Stokes problem from Algorithm 3.1 in the L-shaped domain

	H^1 rate of $\mathbf{u}_j$					L^2 rate of $\mathbf{u}_j$					L^2 rate of p_j				
κ	0.05	0.1	0.2	0.3	0.5	0.05	0.1	0.2	0.3	0.5	0.05	0.1	0.2	0.3	0.5
$j = 7$	1.90	1.84	1.31	0.95	0.54	3.03	3.01	2.99	1.93	1.24	1.89	1.75	1.27	0.95	0.53
$j = 8$	1.95	1.83	1.28	0.95	0.54	3.03	3.01	2.97	1.90	1.18	1.96	1.77	1.27	0.95	0.54
$j = 9$	1.97	1.83	1.27	0.95	0.54	3.01	3.00	2.95	1.89	1.14	1.99	1.78	1.26	0.95	0.54
$j = 10$	1.98	1.82	1.27	0.95	0.54	3.00	2.98	2.93	1.89	1.12	2.00	1.79	1.26	0.95	0.54

Table 12 The CPU time (in seconds) of Algorithm 3.2, Aglorithm 3.1, and the Argyris finite element method

method\ j	Example 5.1 Test case 1					Example 5.1 Test case 2				
	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$
Argyris FEM	7.09	28.57	118.10	492.43	– –	5.27	21.62	87.70	367.76	– –
Algorithm 3.2($k=1$)	0.08	0.39	2.82	15.92	90.94	0.05	0.24	1.13	9.59	58.84
Algorithm 3.1($k=1$)	0.10	0.42	3.01	16.09	96.83	0.08	0.30	1.36	11.37	66.22
Algorithm 3.2($k=2$)	0.88	2.29	9.05	37.72	150.40	0.60	1.90	7.03	28.21	120.95
Algorithm 3.1($k=2$)	0.91	2.74	10.79	43.29	181.42	0.77	2.27	8.25	34.25	147.97

"– –" represents running out of memory

Example 5.3 In this example, we compare the CPU time and the memory usage of the proposed finite element algorithms (Algorithms 3.2 and 3.1) with those of the H^2-conforming Argyris finite element method by solving the biharmonic problem (1.1) in Example 5.1 on the same meshes. The results of the CPU time comparison (in seconds) are shown in Table 12. The results of the memory usage comparison (in GB) are shown in Table 13. All results are tested on MATLAB R2021a in Linux with 16 GB memory and Intel® Core™ i7-6600U processors.

From Table 12, we find that Algorithms 3.2 and 3.1 are much faster than the Argyris finite element method due to the availability of fast Stokes solvers and Poisson solvers. Moreover, Algorithm 3.2 is faster than Algorithm 3.1, since Algorithm 3.1 has one extra Poisson problem to compute. The results in Table 13 indicate that Algorithms 3.2 and 3.1 use much less memory compared with the Argyris finite element method. We notice that there is not much difference in memory usage between Algorithms 3.2 and 3.1, because both algorithms are serial, and the Stokes solver accounts for the maximum memory usage.

Table 13 The memory usage (in GB) of Algorithm 3.2, Aglorithm 3.1, and the Argyris finite element method

method\ j	Example 5.1 Test case 1					Example 5.1 Test case 2				
	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$
Argyris FEM	0.06	0.27	1.00	5.10	– –	0.04	0.13	0.54	3.00	– –
Algorithm 3.2($k=1$)	<0.01	0.01	0.02	0.41	3.03	<0.01	0.01	0.02	0.27	2.20
Algorithm 3.1($k=1$)	<0.01	0.01	0.02	0.46	2.98	<0.01	0.01	0.02	0.27	2.19
Algorithm 3.2($k=2$)	0.01	0.05	0.23	1.07	4.14	<0.01	0.04	0.20	0.80	3.02
Algorithm 3.1($k=2$)	0.01	0.04	0.21	0.93	4.19	<0.01	0.03	0.19	0.80	3.14

"– –" represents running out of memory

6 Conclusion

In this work, we have studied two methods to decompose the biharmonic equation with Dirichlet boundary conditions in Algorithms 3.1 and 3.2. We have investigated the regularity of the solutions for each decomposition in both the Sobolev space and the weighted Sobolev space, demonstrating that the solution of each resulting system is equivalent to that of the original biharmonic problem. We have also designed C^0 finite element algorithms to solve each decoupled system and provided optimal error analysis of the proposed methods on both quasi-uniform and graded meshes. To achieve the optimal convergence rate for the graded finite element approximation, the grading parameter is primarily determined by the polynomial degree of the finite element method and the regularity of the solution to the last Poisson problem in the decomposition. Numerical results are presented to verify the theoretical prediction.

Adding appropriate lower-order terms to problem (1.1) results in the H^2 elliptic problem [20, 27]. Extensions of our methods to such problems may be possible and are left for future work.

Appendix A. Some proofs for Section 3

A.1. Proof of Lemma 3.7

Proof We only present the proof of error estimates of the Taylor-Hood element approximations. The error estimates of the Mini element approximations can be proved similarly. By (3.17),

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_n\|_{[H^1(\Omega)]^2}+\|p-p_n\| \le& C\left(\inf_{\mathbf{v}\in[V_n^k]^2}\|\mathbf{u}-\mathbf{v}\|_{[H^1(\Omega)]^2}+\inf_{q\in S_n^{k-1}}\|p-q\|+\|\mathbf{F}-\mathbf{F}_n\|_{[H^{-1}(\Omega)]^2}\right)\\ \le& Ch^{\min\{k,\alpha,\beta+1\}}+Ch^{\min\{k+1,\beta+1,2\beta\}}\\ \le& Ch^{\min\{k,\alpha,\beta+1\}}(\|\mathbf{u}\|_{[H^{\min\{k+1,\alpha+1,\beta+2\}}]^2}+\|p\|_{\min\{k,\alpha,\beta+2\}}+\|w\|_{H^{\min\{k+1,\beta+1\}}}).\end{aligned}$$

The estimates (3.24b) and (3.24c) can be obtained by

$$\|\mathbf{u}-\mathbf{u}_n\|_{[H^{-l}(\Omega)]^2}=\sup_{g\in[H_0^l(\Omega)]^2}\frac{\langle\mathbf{g},\mathbf{u}-\mathbf{u}_n\rangle}{\|\mathbf{g}\|_{[H^l(\Omega)]^2}},$$

where $l=0,1$. To this end, we will estimate $\langle\mathbf{g},\mathbf{u}-\mathbf{u}_n\rangle$ for $l=0,1$. Subtracting (3.22) with $\mathbf{v}=\mathbf{u}_n$ from (3.19) with $\mathbf{v}=\mathbf{u}$ gives

$$\begin{aligned}\langle\mathbf{g},\mathbf{u}-\mathbf{u}_n\rangle =&(\nabla\mathbf{r},\nabla\mathbf{u})-(\nabla\mathbf{r}_n,\nabla\mathbf{u}_n)-(\mathrm{div}\,\mathbf{u},s)+(\mathrm{div}\,\mathbf{u}_n,s_n)\\ =&(\nabla(\mathbf{u}-\mathbf{u}_n),\nabla\mathbf{r})-(\mathrm{div}\,(\mathbf{u}-\mathbf{u}_n),s)+(\nabla(\mathbf{r}-\mathbf{r}_n),\nabla\mathbf{u}_n)-(\mathrm{div}\,\mathbf{u}_n,s-s_n)\\ =&(\nabla(\mathbf{u}-\mathbf{u}_n),\nabla\mathbf{r})-(\mathrm{div}\,(\mathbf{u}-\mathbf{u}_n),s),\end{aligned}$$

where we have used (3.23a) with $\mathbf{v}=\mathbf{u}_n$ in the last equality. Subtracting (3.15a) with $\mathbf{v}=\mathbf{r}_n$, (3.15b) with $q=s_n$, and (3.19b) with $q=-(p-p_n)$ from the equation above, respectively, we have

$$\begin{aligned}\langle \mathbf{g}, \mathbf{u}-\mathbf{u}_n\rangle =&(\nabla(\mathbf{u}-\mathbf{u}_n), \nabla(\mathbf{r}-\mathbf{r}_n)) - (\text{div }(\mathbf{r}-\mathbf{r}_n), p-p_n)\\ &-(\text{div }(\mathbf{u}-\mathbf{u}_n), s-s_n) + \langle \mathbf{F}-\mathbf{F}_n, \mathbf{r}_n\rangle \\ :=&T_1+T_2+T_3+T_4.\end{aligned} \tag{A.1}$$

For $\mathbf{r}-\mathbf{r}_n$ and $s-s_n$, we have the following estimate

$$\begin{aligned}&\|\mathbf{r}-\mathbf{r}_n\|_{[H^1(\Omega)]^2} + \|s-s_n\| \leq C\left(\inf_{\mathbf{r}_I\in[V_n^k(\Omega)]^2}\|\mathbf{r}-\mathbf{r}_I\|_{[H^1(\Omega)]^2} + \inf_{s_I\in S_n^{k-1}}\|s-s_I\|\right)\\ &\leq Ch^{\min\{k,\alpha,l+1\}}(\|\mathbf{r}\|_{[H^{\min\{k+1,\alpha+1,l+2\}}(\Omega)]^2} + \|s\|_{H^{\min\{k,\alpha,l+1\}}(\Omega)}).\end{aligned} \tag{A.2}$$

Then we have the following estimates for terms T_i, $i=1,\cdots,4$ in (A.1).

$$\begin{aligned}|T_1| \leq&|\mathbf{u}-\mathbf{u}_n|_{[H^1(\Omega)]^2}|\mathbf{r}-\mathbf{r}_n|_{[H^1(\Omega)]^2} \leq Ch^{\min\{k,\alpha,\beta+1\}+\min\{k,\alpha,l+1\}}\|\mathbf{r}\|_{[H^{1+\min\{k,\alpha,l+1\}}(\Omega)]^2}\\ =&Ch^{\min\{2k,k+l+1,k+\alpha,\alpha+l+1,k+\beta+1,\alpha+\beta+1,\beta+l+2,2\alpha\}}\|\mathbf{r}\|_{[H^{1+\min\{k,\alpha,l+1\}}(\Omega)]^2}.\end{aligned}$$

$$|T_2| \leq \|p-p_n\||\mathbf{r}-\mathbf{r}_n|_{[H^1(\Omega)]^2} \leq Ch^{\min\{2k,k+l+1,k+\alpha,\alpha+l+1,k+\beta+1,\alpha+\beta+1,\beta+l+2,2\alpha\}}\|\mathbf{r}\|_{[H^{1+\min\{k,\alpha,l+1\}}(\Omega)]^2}.$$

$$|T_3| \leq |\mathbf{u}-\mathbf{u}_n|_{[H^1(\Omega)]^2}\|s-s_n\| \leq Ch^{\min\{2k,k+l+1,k+\alpha,\alpha+l+1,,k+\beta+1,\alpha+\beta+1,\beta+l+2,2\alpha\}}\|s\|_{H^{\min\{k,\alpha,l+1\}}(\Omega)}.$$

For T_4, we have

$$T_4 = \langle \mathbf{F}-\mathbf{F}_n, \mathbf{r}_n-\mathbf{r}\rangle + \langle \mathbf{F}-\mathbf{F}_n, \mathbf{r}\rangle := T_{41}+T_{42}.$$

$$\begin{aligned}|T_{41}| \leq&\|\mathbf{F}-\mathbf{F}_n\|_{[H^{-1}(\Omega)]^2}\|\mathbf{r}-\mathbf{r}_n\|_{[H^1(\Omega)]^2} \leq Ch^{\min\{k+1,\beta+1,2\beta\}+\min\{k,\alpha,l+1\}}\|\mathbf{r}\|_{[H^{1+\min\{\alpha,l+1\}}(\Omega)]^2}\\ =&Ch^{\min\{2k+1,k+l+2,k+\beta+1,\beta+l+2,\alpha+\beta+1,2\beta+\alpha\}}\|\mathbf{r}\|_{[H^{1+\min\{\alpha,l+1\}}(\Omega)]^2}.\end{aligned}$$

By (3.21) and Lemma 3.6,

$$\begin{aligned}|T_{42}| =&|\langle \mathbf{F}-\mathbf{F}_n, \mathbf{curl}\ \psi\rangle| = |\langle \mathbf{F}-\mathbf{F}_n, \mathbf{curl}\ (\psi-\psi_I)\rangle| \leq \|\mathbf{F}-\mathbf{F}_n\|_{[L^2(\Omega)]^2}\|\mathbf{curl}\ (\psi-\psi_I)\|_{[L^2(\Omega)]^2}\\ \leq&\|\mathbf{F}-\mathbf{F}_n\|_{[L^2(\Omega)]^2}\|\psi-\psi_I\|_{H^1(\Omega)} \leq Ch^{\min\{k,\beta\}+\min\{k,\alpha+1,l+2\}}\|\mathbf{r}\|_{[H^{1+\min\{\alpha,l+1\}}(\Omega)]^2}\\ \leq&Ch^{\min\{2k,k+l+2,k+\beta,\beta+l+2,\alpha+\beta+1\}}\|\mathbf{r}\|_{[H^{1+\min\{\alpha,l+1\}}(\Omega)]^2},\end{aligned}$$

where ψ_I is the nodal interpolation of ψ. It can be verified that

$$|T_4| \leq |T_{41}|+|T_{42}| \leq Ch^{\min\{2k,k+l+2,k+\beta,\beta+l+2,\alpha+\beta+1,\alpha+2\beta\}}\|\mathbf{r}\|_{[H^{1+\min\{\alpha,l+1\}}(\Omega)]^2}.$$

By the regularity (3.20) and the summation of estimates $|T_i|$, $i=1,\cdots,4$, the estimate $\|\mathbf{u}-\mathbf{u}_n\|_{[H^{-l}(\Omega)]^2}$ holds. □

A.2. Proof of Theorem 3.8

Proof Subtracting (3.6) from (2.41) gives

$$(\nabla(\phi-\phi_n),\nabla\psi)=(\text{curl }(\mathbf{u}-\mathbf{u}_n),\psi)\quad\forall\psi\in V_n^k. \tag{A.3}$$

Denote by $\phi_I\in V_n^k$ the nodal interpolation of ϕ. By taking $\epsilon=\phi_I-\phi$, $e=\phi_I-\phi_n$, and $\psi=e$, (A.3) can be written as

$$(\nabla e,\nabla e)=(\nabla\epsilon,\nabla e)+(\text{curl }(\mathbf{u}-\mathbf{u}_n),e)=(\nabla\epsilon,\nabla e)+(\mathbf{u}-\mathbf{u}_n,\mathbf{curl}\ e),$$

which gives

$$\begin{aligned}\|e\|^2_{H^1(\Omega)}\le&\|\epsilon\|_{H^1(\Omega)}\|e\|_{H^1(\Omega)}+\|\mathbf{u}-\mathbf{u}_n\|_{[L^2(\Omega)]^2}\|\mathbf{curl}\ e\|_{[L^2(\Omega)]^2}\\ \le& C(\|\epsilon\|_{H^1(\Omega)}+\|\mathbf{u}-\mathbf{u}_n\|_{[L^2(\Omega)]^2})\|e\|_{H^1(\Omega)},\end{aligned} \tag{A.4}$$

By the triangle inequality, we have

$$\|\phi-\phi_n\|_{H^1(\Omega)}\le\|\epsilon\|_{H^1(\Omega)}+\|e\|_{H^1(\Omega)}\le C\left(\|\epsilon\|_{H^1(\Omega)}+\|\mathbf{u}-\mathbf{u}_n\|_{[L^2(\Omega)]^2}\right). \tag{A.5}$$

Recall that $\phi\in H^{2+\alpha}(\Omega)$. It follows

$$\|\epsilon\|_{H^1(\Omega)}\le Ch^{\min\{k,1+\alpha\}}\|\phi\|_{H^{\min\{k+1,\alpha+2\}}(\Omega)},$$

which together with (3.9) for Algorithm 3.2 and (3.24) for Algorithm 3.1 leads to the conclusion. □

A.3. Proof of Theorem 3.9

Proof Consider the Poisson problem

$$-\Delta v=\phi-\phi_n\ \text{in }\Omega\quad v=0\text{ on }\partial\Omega. \tag{A.6}$$

Then we have

$$\|\phi-\phi_n\|^2=(\nabla(\phi-\phi_n),\nabla v). \tag{A.7}$$

By Subtracting (3.6) from (2.41), it follows

$$(\nabla(\phi-\phi_n),\nabla\psi)=(\text{curl }(\mathbf{u}-\mathbf{u}_n),\psi)\quad\forall\psi\in V_n^k. \tag{A.8}$$

Set $\psi = v_I \in V_n^k$ to be the nodal interpolation of v and subtract (A.8) from (A.7). Then we have

$$\begin{aligned}\|\phi-\phi_n\|^2 =&(\nabla(\phi-\phi_n), \nabla(v-v_I)) + (\text{curl }(\mathbf{u}-\mathbf{u}_n), v_I),\\ =&(\nabla(\phi-\phi_n), \nabla(v-v_I)) + (\text{curl }(\mathbf{u}-\mathbf{u}_n), v_I - v) + (\text{curl }(\mathbf{u}-\mathbf{u}_n), v),\\ =&(\nabla(\phi-\phi_n), \nabla(v-v_I)) + (\mathbf{u}-\mathbf{u}_n, \mathbf{curl}\ (v_I - v)) + (\mathbf{u}-\mathbf{u}_n, \mathbf{curl}\ v),\\ \le&\|\phi-\phi_n\|_{H^1(\Omega)}\|v-v_I\|_{H^1(\Omega)} + \|\mathbf{u}-\mathbf{u}_n\|_{[L^2(\Omega)]^2}\|v-v_I\|_{H^1(\Omega)}\\ &+ \|\mathbf{u}-\mathbf{u}_n\|_{[H^{-\min\{\lfloor\beta\rfloor,1\}}(\Omega)]^2}\|\mathbf{curl}\ v\|_{H^{\min\{\lfloor\beta\rfloor,1\}}(\Omega)},\end{aligned}$$

where $\lfloor\cdot\rfloor$ represents the floor function.

The regularity result [32, 33] of the Poisson problem (A.7) gives

$$\|v\|_{H^{\min\{1+\beta,2\}}(\Omega)} \le C\|\phi-\phi_n\|_{H^{\min\{\beta-1,0\}}(\Omega)} \le C\|\phi-\phi_n\|, \tag{A.9}$$

where $\beta < \frac{\pi}{\omega}$. From (3.8), we have

$$\|v-v_I\|_{H^1(\Omega)} \le Ch^{\min\{\beta,1\}}\|v\|_{H^{\min\{1+\beta,2\}}(\Omega)}.$$

For Algorithm 3.2, we have the following result by (3.9). Recall that $\beta < \frac{\pi}{\omega}$, if $\omega > \pi$ we have $\lfloor\beta\rfloor = 0$ and

$$\|\mathbf{u}-\mathbf{u}_n\|_{[H^{-\min\{\lfloor\beta\rfloor,1\}}(\Omega)]^2} = \|\mathbf{u}-\mathbf{u}_n\|_{[L^2(\Omega)]^2} \le Ch^{2\alpha}; \tag{A.10}$$

if $\omega < \pi$, we have $\lfloor\beta\rfloor = 1$ and

$$\|\mathbf{u}-\mathbf{u}_n\|_{[H^{-\min\{\lfloor\beta\rfloor,1\}}(\Omega)]^2} \le Ch^{\min\{2k,k+2,k+\alpha,\alpha+2,2\alpha\}}. \tag{A.11}$$

For $\omega \in (0, 2\pi)\setminus\{\pi\}$, (A.10) and (A.11) imply that

$$\|\mathbf{u}-\mathbf{u}_n\|_{[H^{-\min\{\lfloor\beta\rfloor,1\}}(\Omega)]^2} \le Ch^{\min\{2k,k+2,k+\alpha,\alpha+2,2\alpha\}}. \tag{A.12}$$

Thus, we have by (3.25), (3.9), and (A.12)

$$\begin{aligned}\|\phi-\phi_n\|^2 \le& Ch^{\min\{k+1,\alpha+2,k+\beta,2\alpha+\beta\}}\|v\|_{H^{\min\{1+\beta,2\}}(\Omega)} + Ch^{\min\{2k,k+2,k+\alpha,\alpha+2,2\alpha\}}\|v\|_{H^{\min\{1+\beta,2\}}(\Omega)}\\ \le& Ch^{\min\{k+1,\alpha+2,2\alpha\}}\|v\|_{H^{\min\{1+\beta,2\}}(\Omega)}.\end{aligned} \tag{A.13}$$

By (A.9) and (A.13), the estimate (3.27) holds.

Similarly, for Algorithm 3.1 we have

$$\begin{aligned}\|\phi-\phi_n\|^2 \le& Ch^{\min\{k+1,\alpha+2,\beta+3,k+\beta,2\alpha+\beta\}}\|v\|_{H^{\min\{1+\beta,2\}}(\Omega)}\\ &+ Ch^{\min\{2k,k+2,k+\beta,\alpha+2,\beta+3,2\alpha\}}\|v\|_{H^{\min\{1+\beta,2\}}(\Omega)}\\ \le& Ch^{\min\{k+1,\alpha+2,\beta+3,2\alpha\}}\|v\|_{H^{\min\{1+\beta,2\}}(\Omega)}.\end{aligned} \tag{A.14}$$

By (A.9) and (A.14), the estimate (3.28) holds. □

Appendix B. Some proofs for Section 4

B.1. Proof of Lemma 4.6

Proof In $\kappa_{Q_i} = 2^{-\frac{\theta}{a_i}}$, we take $a_i = \frac{\alpha_0}{\beta_0} b_i$, where $b_i < \min\{\beta_0^i, \frac{\beta_0}{\alpha_0}\alpha_0^i\}$ and $b_i \le \theta'$. Then we have

$$\kappa_{Q_i} = 2^{-\frac{\theta}{a_i}} = 2^{-\frac{\frac{\beta_0}{\alpha_0}\theta}{b_i}}.$$

By (4.7) and the interpolation error estimates in Lemma 4.3 under the regularity result in (4.6), we have

$$\|w - w_n\|_{H^1(\Omega)} \le C\|w - w_I\|_{H^1(\Omega)} \le Ch^{\theta'}\|w\|_{\mathcal{K}^{\theta'+1}_{\mathbf{b}+1}(\Omega)}.$$

Consider the Poisson problem

$$-\Delta v = w - w_n \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega. \tag{B.1}$$

Then we have

$$\|w - w_n\|^2 = (\nabla(w - w_n), \nabla v). \tag{B.2}$$

Subtracting (3.3) from (2.16) yields the Galerkin orthogonality,

$$(\nabla(w - w_n), \nabla\psi) = 0 \quad \forall\psi \in V_n^k. \tag{B.3}$$

Setting $\psi = v_I \in V_n^k$ the nodal interpolation of v and subtract (B.3) from (B.2) gives

$$\|w - w_n\|^2 = (\nabla(w - w_n), \nabla(v - v_I)) \le \|w - w_n\|_{H^1(\Omega)}\|v - v_I\|_{H^1(\Omega)}. \tag{B.4}$$

Similarly, the solution $v \in \mathcal{K}^2_{\mathbf{b}'+1}(\Omega)$ satisfies the regularity estimate

$$\|v\|_{K^2_{\mathbf{b}'+1}(\Omega)} \le C\|w - w_n\|_{K^0_{\mathbf{b}'-1}(\Omega)} \le C\|w - w_n\|, \tag{B.5}$$

where the ith entry of $\mathbf{b}'$ satisfying $b_i' = \min\{b_i, 1\}$. By Lemma 4.3 again, we have the interpolation error

$$\|v - v_I\|_{H^1(\Omega)} \le Ch^{\min\{\theta',1\}}\|v\|_{K^2_{\mathbf{b}'+1}(\Omega)}. \tag{B.6}$$

The L^2 error estimate in (4.8) can be obtained by combining (B.4), (B.5), and (B.6).□

B.2. Proof of Theorem 4.8

Proof For Algorithm 3.2, by (3.16) and the interpolation error estimates in Lemma 4.3, along with the regularity result in Lemma 4.4, the estimate (4.13) holds.

For Algorithm 3.1, by (3.17) with the estimate (4.9), the interpolation error estimates in Lemma 4.3, it follows

$$\begin{aligned}&\|\mathbf{u}-\mathbf{u}_n\|_{[H^1(\Omega)]^2}+\|p-p_n\|\leq Ch^{\min\{\theta,\theta'+1\}}(\|\mathbf{u}\|_{[\mathcal{K}_{\mathbf{a}+1}^{\min\{\theta,\theta'+1\}+1}(\Omega)]^2}\\&+\|p\|_{\mathcal{K}_{\mathbf{a}}^{\min\{\theta,\theta'+1\}}(\Omega)})+Ch^{\min\{\theta'+1,2\theta'\}}\|w\|_{\mathcal{K}_{\mathbf{b}+1}^{\theta'+1}(\Omega)}\quad\leq Ch^{\min\{\theta,\theta'+1\}}D_1.\end{aligned}$$

Here, we have used the fact that if $\omega > \pi$, $\theta \leq \theta' < 2\theta'$. Note that $\theta' = \min\left\{\max\left\{\frac{\beta_0}{\alpha_0}\theta, \beta\right\}, k\right\}$, so if $\theta' = k$, then $\theta \leq k = \theta'$; otherwise

$$\theta' \geq \max\left\{\frac{\beta_0}{\alpha_0}\theta, \beta\right\} \geq \frac{\beta_0}{\alpha_0}\theta > \theta,$$

where we have used Lemme 2.1.

If $\omega < \pi$, by taking $1 < b_i = \frac{\beta_0}{\alpha_0}a_i < \beta_0 \leq \beta_0^i$, it follows $\theta' \geq b_i > 1$, so that $\theta' + 1 < 2\theta'$. Thus, the estimate (4.14) holds. $\square$

B.3. Proof of Theorem 4.9

Proof We will only prove (4.17) for the Taylor-Hood method, but all other cases can be proved similarly. By taking $\mathbf{v} = \mathbf{u} - \mathbf{u}_n$, $q = p - p_n$ in (3.19), we have

$$\|\mathbf{u}-\mathbf{u}_n\|_{[(\mathcal{K}_{\mathbf{b}}^l(\Omega))^*]^2} = \sup_{g\in(\mathcal{K}_{\mathbf{b}}^l(\Omega))]^2}\frac{\langle\mathbf{g},\mathbf{u}-\mathbf{u}_n\rangle}{\|\mathbf{g}\|_{[(\mathcal{K}_{\mathbf{b}}^l(\Omega))]^2}},$$

where $l = 0, 1$. Let $(\mathbf{r}_n, s_n)$ be the Taylor-Hood solution of (3.18), then it follows

$$\langle\mathbf{g},\mathbf{u}-\mathbf{u}_n\rangle = T_1 + T_2 + T_3 + T_4,$$

where $T_i, i = 1, \cdots, 4$ have the same expressions as those in Lemma 3.7. For $\mathbf{r}_n$ and s_n, we have the following estimate in the weighted Sobolev space

$$\begin{aligned}&\|\mathbf{r}-\mathbf{r}_n\|_{[H^1(\Omega)]^2}+\|s-s_n\|\leq C\left(\inf_{\mathbf{r}_I\in[V_n^k(\Omega)]^2}\|\mathbf{r}-\mathbf{r}_I\|_{[H^1(\Omega)]^2}+\inf_{s_I\in S_n^{k-1}}\|s-s_I\|\right)\\&\leq Ch^{\min\{\theta,l+1\}}(\|\mathbf{r}\|_{[\mathcal{K}_{\mathbf{a}'+1}^{l+2}(\Omega)]^2}+\|s\|_{\mathcal{K}_{\mathbf{a}'}^{l+1}(\Omega)}).\end{aligned}\tag{B.7}$$

Here we have

$$
\begin{aligned}
|T_j| \leq & Ch^{\min\{\theta,\theta'+1\}+\min\{\theta,l+1\}}(\|\mathbf{r}\|_{[\mathcal{K}^{l+2}_{\mathbf{a}'+1}(\Omega)]^2} + \|s\|_{\mathcal{K}^{l+1}_{\mathbf{a}'}(\Omega)}) \\
= & Ch^{\min\{2\theta,\theta+l+1,\theta+\theta'+1,\theta'+l+2\}}(\|\mathbf{r}\|_{[\mathcal{K}^{l+2}_{\mathbf{a}'+1}(\Omega)]^2} + \|s\|_{\mathcal{K}^{l+1}_{\mathbf{a}'}(\Omega)}),
\end{aligned}
$$

where $j = 1, 2, 3$.

Note that

$$T_4 = T_{41} + T_{42} = \langle \mathbf{F} - \mathbf{F}_n, \mathbf{r}_n - \mathbf{r}\rangle + \langle \mathbf{F} - \mathbf{F}_n, \mathbf{r}\rangle.$$

By (4.9) and (B.7), we have

$$
\begin{aligned}
|T_{41}| \leq & \|\mathbf{F} - \mathbf{F}_n\|_{[H^{-1}(\Omega)]^2}\|\mathbf{r} - \mathbf{r}_n\|_{[H^1(\Omega)]^2} \leq Ch^{\min\{\theta'+1,2\theta'\}+\min\{\theta,l+1\}}\|\mathbf{r}\|_{[\mathcal{K}^{l+2}_{\mathbf{a}'+1}(\Omega)]^2} \\
= & Ch^{\min\{\theta+\theta'+1,\theta'+l+2,2\theta'+l+1,\theta+2\theta'\}}\|\mathbf{r}\|_{[\mathcal{K}^{l+2}_{\mathbf{a}'+1}(\Omega)]^2}.
\end{aligned}
$$

We reformulate $\kappa_{Q_i} = 2^{-\frac{\theta}{a_i}} = 2^{-\frac{\theta_1}{1+a_i'}}$, where $\theta_1 = \frac{1+a_i'}{a_i}\theta$. By Lemma 4.3 for $\psi \in \mathcal{K}^{l+3}_{\mathbf{a}'+2}(\Omega)$ satisfying (3.21), we have

$$\|\psi - \psi_I\|_{H^1(\Omega)} \leq Ch^{\min\{k,\theta_1,l+2\}}\|\mathbf{r}\|_{[\mathcal{K}^{l+2}_{\mathbf{a}'+1}(\Omega)]^2}, \tag{B.8}$$

where $\theta_1 \geq (1 + \frac{1}{a_i})\theta \geq \theta + 1$ and ψ_I is the nodal interpolation of ψ.

By (4.9) and (B.8), we have

$$
\begin{aligned}
|T_{42}| \leq & \|\mathbf{F} - \mathbf{F}_n\|_{[L^2(\Omega)]^2}\|\psi - \psi_I\|_{H^1(\Omega)} \leq Ch^{\theta'+\min\{k,\theta+\theta_1',l+2\}}\|\mathbf{r}\|_{[\mathcal{K}^{l+2}_{\mathbf{a}'+1}(\Omega)]^2} \\
\leq & Ch^{\min\{k+\theta',\theta'+l+2,\theta+\theta'+\theta_1'\}}\|\mathbf{r}\|_{[\mathcal{K}^{l+2}_{\mathbf{a}'+1}(\Omega)]^2},
\end{aligned}
$$

where $\theta_1' = \min_i\{\frac{1}{a_i'}\theta\} \geq 1$. It can be verified that

$$|T_4| \leq |T_{41}| + |T_{42}| \leq Ch^{\min\{k+\theta',\theta+\theta'+1,\theta'+l+2,2\theta'+l+1,\theta+2\theta'\}}\|\mathbf{r}\|_{[\mathcal{K}^{l+2}_{\mathbf{a}'+1}(\Omega)]^2}.$$

By the regularity (4.15) and the summation of estimates $|T_i|$, $i = 1, \cdots, 4$, and $\theta < 2\theta'$, we have the error estimate

$$\|\mathbf{u} - \mathbf{u}_n\|_{[(\mathcal{K}^1_{\mathbf{b}}(\Omega))^*]^2} \leq Ch^{\min\{2\theta,\theta+l+1,k+\theta',\theta+\theta'+1,\theta'+l+2\}}. \tag{B.9}$$

Recall that $k \geq 1$, $\theta \leq k$, and when $\omega > \pi$, we have $\theta < \theta'$, then it follows

$$\theta + 1 \leq k + \theta', \tag{B.10}$$

and when $\omega < \pi$, we have $\theta' > 1$, so the inequality (B.10) still holds. The estimates in (4.17) follow from (B.9) with the fact (B.10). □

B.4. Proof of Theorem 4.11

Proof Denote by $\phi_I \in V_n^k$ the nodal interpolation of ϕ. Similar to Theorem 3.8, we have

$$\|\phi - \phi_n\|_{H^1(\Omega)} \leq C \left(\|\phi - \phi_I\|_{H^1(\Omega)} + \|\mathbf{u} - \mathbf{u}_n\|_{[L^2(\Omega)]^2}\right) \tag{B.11}$$

Recall that $\phi \in \mathcal{K}^{m+2}_{\mathbf{a}+2}(\Omega) = \mathcal{K}^{(m+1)+1}_{(\mathbf{a}+1)+1}(\Omega)$ with $m \geq k$. Note that $\kappa_{Q_i} = 2^{-\frac{\theta}{a_i}} = 2^{-\frac{\theta_1}{1+a_i}}$, where $\theta_1 = \frac{1+a_i}{a_i}\theta = \theta + \frac{1}{a_i}\theta \geq \theta + 1 \geq k$. By Lemma 4.3, we have

$$\|\phi - \phi_I\|_{H^1(\Omega)} \leq Ch^{\min\{k,\theta_1\}} = Ch^k \|\phi\|_{\mathcal{K}^{k+2}_{\mathbf{a}+2}(\Omega)}. \tag{B.12}$$

For θ given in (4.21), Theorem 4.9 indicates for Algorithm 3.2,

$$\|\mathbf{u} - \mathbf{u}_n\|_{[L^2(\Omega)]^2} \leq Ch^{\min\{\max\{2\alpha,2(k-1)\},\max\{\alpha+1,k\},k+1\}} D_2. \tag{B.13}$$

Plugging (B.12) and (B.13) into (B.11), the estimate (4.22) holds.

For Algorithm 3.1, we have

$$\|\mathbf{u} - \mathbf{u}_n\|_{[L^2(\Omega)]^2} \leq Ch^{\min\left\{\max\{2\alpha,2(k-1)\},\max\{\alpha+1,k\},\frac{\beta_0}{\alpha_0}\max\{k-1,\alpha\}+2,k+1\right\}} D_1. \tag{B.14}$$

By plugging (B.12) and (B.14) into (B.11), it follows the estimate

$$\|\phi - \phi_n\|_{H^1(\Omega)} \leq Ch^{\min\left\{k,\max\left\{\frac{\beta_0}{\alpha_0}(k-1)+2,\beta+2\right\}\right\}}.$$

If $\omega > \pi$, we have $\beta_0 > \alpha_0$ and

$$\frac{\beta_0}{\alpha_0}(k-1) + 2 > k + 1 > k, \quad \forall k \geq 1.$$

Then case (ii) in (4.23a) holds.

If $\omega < \pi$, we have $\beta > 1$ and $\beta + 2 \geq 3$. Therefore, case (i) in (4.23a) holds for $k \leq 3$. For $k = 4$, we have $\frac{\beta_0}{\alpha_0} > \frac{2}{3}$ as shown in Fig. 2(c). Therefore, it follows

$$\frac{\beta_0}{\alpha_0}(k-1) + 2 > \frac{2}{3}(k-1) + 2 = 4.$$

Thus, (4.23a) also holds. □

B.5. Proof of Theorem 4.12

Proof Set $\psi = v_I \in V_n^k$ the nodal interpolation of v of the Poisson problem (A.7). Similar to Theorem 3.9, we have

$$\begin{aligned}\|\phi-\phi_n\|^2 \leq & \|\phi-\phi_n\|_{H^1(\Omega)}\|v-v_I\|_{H^1(\Omega)} + \|\mathbf{u}-\mathbf{u}_n\|_{[L^2(\Omega)]^2}\|v-v_I\|_{H^1(\Omega)} \\ & + \|\mathbf{u}-\mathbf{u}_n\|_{[(\mathcal{K}^1_{\mathbf{b}}(\Omega))^*]^2}\|\mathbf{curl}\ v\|_{[\mathcal{K}^1_{\mathbf{b}}(\Omega)]^2} := T_1+T_2+T_3.\end{aligned} \tag{B.15}$$

Based on the results in [7], the solution $v \in \mathcal{K}^2_{\mathbf{b}'+1}(\Omega)$ satisfies the regularity estimate

$$\|v\|_{K^2_{\mathbf{b}'+1}(\Omega)} \leq C\|\phi-\phi_n\|, \tag{B.16}$$

where the ith entry of $\mathbf{b}'$ is given by $b_i' = \min\{b_i, 1\}$ with $b_i < \frac{\pi}{\omega_i}$. If $\omega > \pi$, we have

$$\theta' \geq \theta \geq \frac{k+1}{2} \geq 1,$$

so by the interpolation error (4.3),

$$\|v-v_I\|_{H^1(\Omega)} \leq Ch^{\min\{\theta',1\}}\|v\|_{K^2_{\mathbf{b}'+1}(\Omega)} = Ch\|v\|_{K^2_{\mathbf{b}'+1}(\Omega)}. \tag{B.17}$$

If $\omega < \pi$, the interpolation error (B.17) is obvious since $v \in H^2(\Omega)$.

For Algorithm 3.2, we have the following estimate for each T_i, $i = 1, 2, 3$. By Theorem 4.11 and (B.17), it follows

$$T_1 = \|\phi-\phi_n\|_{H^1(\Omega)}\|v-v_I\|_{H^1(\Omega)} \leq Ch^{k+1}\|v\|_{K^2_{\mathbf{b}'+1}(\Omega)}.$$

By Theorem 4.9 and (B.17), it follows

$$T_2 = \|\mathbf{u}-\mathbf{u}_n\|_{[L^2(\Omega)]^2}\|v-v_I\|_{H^1(\Omega)} \leq Ch^{k+2}\|v\|_{K^2_{\mathbf{b}'+1}(\Omega)}.$$

Again, by Theorem 4.9, especially with $\theta \geq \frac{k+1}{2}$, it follows

$$T_3 \leq C\|\mathbf{u}-\mathbf{u}_n\|_{[(\mathcal{K}^1_{\mathbf{b}'}(\Omega))^*]^2}\|v\|_{\mathcal{K}^2_{\mathbf{b}'+1}(\Omega)} \leq Ch^{k+1}\|v\|_{K^2_{\mathbf{b}'+1}(\Omega)},$$

Thus, the regularity estimate (B.16) and the summation of T_i, $i = 1, 2, 3$ give the estimate (4.25).

For Algorithm 3.1, we have the following estimate for T_i. By Theorem 4.11 and (B.17), it follows

$$T_1 \leq Ch^{\min\left\{k+1, \max\left\{\frac{\beta_0}{\alpha_0}(k-1)+2, \beta+2\right\}+1\right\}}\|v\|_{K^2_{\mathbf{b}'+1}(\Omega)}.$$

By Theorem 4.9 and (B.17), it follows

$$T_2 \leq Ch^{\min\left\{k+2,\max\left\{\frac{\beta_0}{\alpha_0}(k-1)+2,\beta+2\right\}+2\right\}}\|v\|_{K^2_{\mathbf{b}'+1}(\Omega)}.$$

Again, it follows from Theorem 4.9

$$T_3 \leq Ch^{\min\left\{k+1,\max\left\{\frac{\beta_0}{\alpha_0}(k-1)+2,\beta+2\right\}+1\right\}}\|v\|_{K^2_{\mathbf{b}'+1}(\Omega)}.$$

Thus, the regularity estimate (B.16) and the summation of T_i, $i = 1, 2, 3$ again give the estimate (4.26). □

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