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A C^0 Finite Element Algorithm for the Sixth Order Problem With Simply Supported Boundary Conditions

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ABSTRACT

In this paper, we investigate a sixth-order elliptic equation with the simply supported boundary conditions in a polygonal domain. We propose a new method that decouples the sixth-order problem into a system of second-order equations. Unlike the direct decomposition, which yields three Poisson problems but is restricted to polygonal domains with the largest interior angle no more than $\pi/2$, we rigorously analyze and construct extra Poisson problems to confine the solution into the same function space as that of the original sixth-order problem. Consequently, the proposed method can be applied to general polygonal domains. In turn, we also present a C^0 finite element algorithm to discretize the new resulting system and establish optimal error estimates for the numerical solution on quasi-uniform meshes. Finally, numerical experiments are performed to validate the theoretical findings.

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1 | Introduction

Consider the sixth-order elliptic problem, also known as the tri-harmonic problem

$$-\Delta^3 u = f \quad \text{in } \Omega, \qquad u = \Delta u = \Delta^2 u = 0 \quad \text{on } \partial\Omega \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a polygonal domain and $f \in H^{-1}(\Omega)$. The boundary conditions in (1.1) are commonly referred to as simply supported boundary conditions [1]. The sixth-order partial differential equations (PDEs) arise from various mathematical models, including applications in differential geometry [2], thin film equations [3], and the phase field crystal model [4–7]. The conforming finite element approximation for (1.1) necessitates H^3 conforming finite elements, typically involving intricate constructions of the finite element space and the variational formulation [8–11]. Recently, a nonconforming H^3 finite element was proposed in [12], where the element is composed of H^1 conforming finite elements and additional bubble functions. C^0 interior penalty discontinuous Galerkin (IPDG) and C^1-IPDG methods were proposed in [13] for the sixth-order elliptic equations with clamped boundary conditions. To balance the weak continuity and the complexity associated with choosing penalty parameters, a family of $\mathcal{P}_m$ interior nonconforming finite element methods was proposed in [14]. Additionally, a mixed finite element method was introduced in [1], based on low-order H^1 conforming finite elements, with an optimal error estimate under an appropriate regularity assumption.

Hengguang Li and Peimeng Yin contributed equally to this study.

The direct mixed finite element method, employing C^0 finite elements, offers an appealing approach for addressing high-order elliptic problems, such as the biharmonic problem [15–18] and the sixth-order problem (1.1). This is primarily due to the boundary conditions, which facilitate the derivation of three entirely decoupled Poisson equations. This suggests that a plausible numerical solution could be attained by simply employing a finite element Poisson solver within the mixed formulation. However, while the implementation of the mixed finite element method is straightforward, its solution may not always be reliable, as the solution obtained from the Poisson problem might reside in a different Sobolev space compared to that of the original sixth-order problem (1.1). This discrepancy is evident in the fact that the solution to the Poisson problem typically belongs to $H^1(\Omega)$, whereas that of the sixth-order problem (1.1) usually belongs to $H^3(\Omega)$. This phenomenon was identified in the context of the biharmonic equation with Navier boundary conditions, known as the Sapongyan paradox [19, 20]. To confine the solution of the Poisson problem to $H^2(\Omega)$, an additional Poisson problem needs to be solved [15], particularly when the polygonal domain features a re-entrant corner. For the sixth-order problem (1.1), achieving confinement of the solution to $H^3(\Omega)$ is not a trivial task.

The direct mixed formulation, which decomposes the problem into three Poisson equations, actually defines a weak solution in a larger function space compared to that of Equation (1.1). This mismatch in function spaces does not impact the solution in a polygonal domain where the largest interior angle is no more than $\pi/2$. However, when the largest interior angle exceeds $\pi/2$, the direct mixed method allows for additional singular functions, leading to a solution different from that of Equation (1.1). To confine the solution to the correct function spaces, we propose a modified mixed formulation aiming at eliminating the singular functions. More specifically, we first rigorously establish that the space of the singular functions, or equivalently, their image space under the Laplace operator, is finite-dimensional. In particular, the dimension of the singular function space associated with a corner depends on the corresponding interior angle: it is 0 if the angle lies in $(0, \pi/2]$, 1 if in $(\pi/2, \pi)$, 2 if in $(\pi, 3\pi/2]$, and 3 if in $(3\pi/2, 2\pi)$. Subsequently, we identify a basis for the singular function space, or equivalently, its image space. Finally, we formulate the modified mixed formulation by removing the solution component that resides in the singular function space. The resulting formulation is shown to be well-posed, and the solution is equivalent to the original problem.

In turn, we introduce a numerical algorithm to solve the proposed mixed formulation, utilizing piecewise linear C^0 finite elements on quasi-uniform meshes. Meanwhile, we conduct an error analysis on the finite element approximations for both the auxiliary functions and the solution u. For the auxiliary functions, the errors in the H^1 norm are standard and have a convergence rate $h^{\min\{\frac{\pi}{\omega},1\}}$, where ω is the largest interior angle of the polygonal domain; the L^2 error estimates can be obtained using the duality argument. For the approximation to the solution u, the error in the H^1 norm is bounded by (i) the H^1 interpolation error of the solution u; (ii) the H^{-1} error for the auxiliary functions; and (iii) the H^1 errors and the weighted L^2 error for the approximations to the additional intermediate Poisson problems that confine the solution to the correct function space. Depending on the largest interior angle, the convergence rate for the H^1 error of the numerical solution is dominated by either the degree of the polynomials or the singularity of the intermediate functions.

In summary, we propose a C^0 finite element algorithm that reduces the sixth-order problem with simply supported boundary conditions to a system of second-order equations. The key contributions of this work are outlined as follows:

- Compared to existing penalty methods and nonconforming approaches, the proposed method is simple and intuitive in its formulation, and a plausible numerical solution can be obtained using only a standard C^0 finite element Poisson solver.
- The direct mixed formulation, which decomposes the original problem into three Poisson problems, fails to maintain equivalence with the original problem when the largest interior angle exceeds $\pi/2$. In contrast, by carefully confining the intermediate functions to the appropriate function space, the proposed method remains valid for general polygonal domains, regardless of whether any interior angle exceeds $\pi/2$ or not.
- We rigorously derive optimal error estimates for the proposed method on quasi-uniform meshes using C^0 linear finite element polynomials.
- Based on the largest interior angle of the domain, we conduct numerical tests to compare the solutions obtained from the direct mixed finite element method and the proposed method. In addition, we evaluate the convergence rate of the proposed method.

The rest of the paper is organized as follows: In Section 2, according to the general regularity theory for second-order elliptic equations [21–25], we introduce the weak solution of the sixth-order problem (1.1). Additionally, we discuss the orthogonal space of the image of the operator $-\Delta$ in $H_0^1(\Omega)$ and identify basis functions of this space. We then propose a modified mixed formulation and demonstrate the equivalence of the solution to that of the original sixth-order problem. In Section 3, we present the finite element algorithm and derive error estimates on quasi-uniform meshes for both the solution u and the auxiliary functions. Finally, in Section 4, we present numerical test results to validate the theory.

Throughout the paper, the generic constant $C > 0$ in our estimates may vary across different occurrences. Its value depends on the computational domain but remains independent of the functions involved or the mesh level in the finite element algorithms.

2 | The Sixth Order Problem

2.1 | Well-Posedness of the Solution

Denote by $H^m(\Omega)$, $m \geq 0$, the Sobolev space consisting of functions whose ith derivatives are square integrable for $0 \leq i \leq m$. Let $L^2(\Omega) := H^0(\Omega)$. If m is not an integer, then it defines the fractional Sobolev space. Denote by $\mathcal{D}(\Omega)$ the space of infinitely differentiable functions in Ω with compact support.

We define $H_0^s(\Omega)$ to be the closure of $\mathcal{D}(\Omega)$ in $H^s(\Omega)$. Recall that $H_0^s(\Omega) \subset H^s(\Omega)$ for $0 < s \le 1$ is the subspace consisting of functions with zero traces on the boundary $\partial\Omega$ [26]. We shall denote the norm $\|\cdot\|_{L^2(\Omega)}$ by $\|\cdot\|$ when there is no ambiguity about the underlying domain. Recall that for $D \subseteq \mathbb{R}^d$, the fractional order Sobolev space $H^s(D)$ consists of distributions v in D satisfying

$$\|v\|_{H^s(D)}^2 := \|v\|_{H^m(D)}^2 + \sum_{|\alpha|=m} \int_D \int_D \frac{|\partial^\alpha v(x) - \partial^\alpha v(y)|^2}{|x-y|^{d+2t}} dxdy < \infty,$$

where $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}_{\ge 0}^d$ is a multi-index such that $\partial^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}$ and $|\alpha| = \sum_{i=1}^d \alpha_i$.

We define the space

$$V = \{\phi \mid \phi \in H^3(\Omega), \phi|_{\partial\Omega} = 0,\ \Delta\phi|_{\partial\Omega} = 0\} \tag{2.1}$$

then the variational formulation for Equation (1.1) is to find $u \in V$ such that,

$$a(u, \phi) := \int_\Omega \nabla\Delta u \cdot \nabla\Delta\phi dx = \int_\Omega f\phi dx = (f, \phi), \quad \forall \phi \in V \tag{2.2}$$

For (2.2), we have the following result:

Lemma 2.1. *Given $f \in H^{-1}(\Omega)$ for the variational formulation (2.2), there exists at most one solution in V.*

Proof. We postpone the proof of the existence of the solution to Theorem 2.17. Assume that (2.2) has two solutions u_1 and u_2 in V. Let $\delta u = u_1 - u_2$. Then we have

$$a(\delta u, \phi) = 0, \quad \phi \in V \tag{2.3}$$

Note that $\delta u \in V$ implies $\Delta\delta u \in H_0^1(\Omega)$. In addition $\delta u \in H^2(\Omega) \cap H_0^1(\Omega)$. Then, by the Poincaré-type inequality,

$$\|\nabla\Delta\delta u\| \ge C_0\|\Delta\delta u\|_{H^1(\Omega)} \ge C_0\|\Delta\delta u\| \ge C\|\delta u\|_{H^2(\Omega)},$$

where ([21], Theorem 2.2.3) has been used in the last inequality. By setting $\phi = \delta u$ in (2.3), it follows

$$0 = a(\delta u, \delta u) = \|\nabla\Delta\delta u\|^2 \ge C\|\delta u\|_{H^2(\Omega)} = 0.$$

Thus $\delta u = 0$, which implies $u_1 = u_2$ in $H^2(\Omega)$, and therefore $u_1 = u_2$ in V. □

2.2 | The Direct Mixed Formulation

Intuitively, we can decouple (1.1) into a system of three Poisson problems by introducing auxiliary functions w and v, satisfying:

$$\begin{cases} -\Delta w = f & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega; \end{cases} \quad \begin{cases} -\Delta v = w & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega; \end{cases}$$
$$\text{and} \quad \begin{cases} -\Delta \bar{u} = v & \text{in } \Omega, \\ \bar{u} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

We refer to (2.4) as the direct mixed formulation. Note that numerical solvers for the Poisson problems (2.4) are readily available, while numerical approximation of the sixth-order problem (1.1) is generally a daunting task. The weak formulation of (2.4) is to find $\bar{u}, v, w \in H_0^1(\Omega)$ such that

$$A(w, \phi) = (f, \phi), \quad \forall \phi \in H_0^1(\Omega) \tag{2.5a}$$

$$A(v, \psi) = (w, \psi), \quad \forall \psi \in H_0^1(\Omega) \tag{2.5b}$$

$$A(\bar{u}, \tau) = (v, \tau), \quad \forall \tau \in H_0^1(\Omega) \tag{2.5c}$$

where

$$A(\phi, \psi) = \int_\Omega \nabla\phi \cdot \nabla\psi dx.$$

Assuming that the source term f in (2.5) and (2.2) satisfies $f \in H^{-1}(\Omega) \subset V^*$, the solutions $\bar{u}, v, w$ of the Poisson problems in (2.5) are well-defined [27]. The important question is whether the solution $\bar{u}$ in (2.5) is the same as the solution u in (2.2).

To address this question, it is imperative to delve into the solution structure of the Poisson problem within a polygonal domain. This exploration will be undertaken in the subsequent subsection.

2.3 | Image of the Laplace Operator in $H_0^1(\Omega)$ and Its Orthogonal Space

Assume that the polygonal domain Ω has at most one interior angle greater than $\frac{\pi}{2}$. Let ω be the largest interior angle with the vertex Q. Without loss of generality, we set Q as the origin and represent polar coordinates centered at the vertex Q as (r, θ), where the interior angle ω is spanned by two half lines $\theta = 0$ and $\theta = \omega$. We construct a sector $K_\omega^R \subset \Omega$ at Q with radius $R > 0$ as

$$K_\omega^R = \{(r\cos\theta, r\sin\theta) \in \Omega \mid 0 < r < R, 0 < \theta < \omega\}.$$

A sketch drawing of the domain Ω is depicted in Figure 1.

To begin with, we introduce a general Poisson problem

$$-\Delta z = g \text{ in } \Omega, \qquad z = 0 \text{ on } \partial\Omega \tag{2.6}$$

Recall the space V in (2.1). For any function $\phi \in V$, it can be verified that $-\Delta\phi \in H_0^1(\Omega)$. Then we have the following result.

Lemma 2.2. *The mapping$-\Delta : V \to H_0^1(\Omega)$ is injective and has a closed range, where the subspace V is given in (2.1).*

Proof. Let z_1, z_2 be functions in $V \subset H_0^1(\Omega)$ satisfying $\Delta z_1 = \Delta z_2$. Then the function $g = -\Delta z_1 = -\Delta z_2 \in H_0^1(\Omega)$. By the Lax–Milgram Theorem for the Poisson problem (2.6), it follows

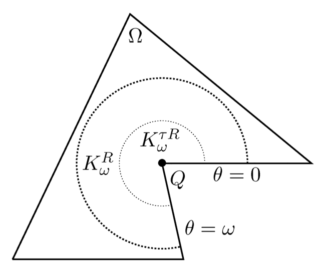


FIGURE 1 | Domain Ω containing a reentrant corner.

$z_1 = z_2$ in $H_0^1(\Omega)$, and hence $z_1 = z_2$ in V, demonstrating the injective nature of the mapping.

Denote the image of the mapping by $\mathcal{M} \subset H_0^1(\Omega)$. Consider a sequence $\{g_i\}_{i=1}^{\infty}$ in $\mathcal{M}$ satisfying $g_i := -\Delta z_i \to g$ for $z_i \in H^3(\Omega)$, which implies that $g_i \in \mathcal{M}$ is Cauchy and $g \in H_0^1(\Omega)$. We now show $\mathcal{M}$ is closed, namely, $g \in \mathcal{M}$. By the regularity result for the elliptic equation, it holds

$$\|z_m - z_n\|_{H^3(\Omega)} \leq C\|g_m - g_n\|_{H^1(\Omega)} \tag{2.7}$$

which implies $\{z_i\}_{i=1}^{\infty}$ is also Cauchy in V. Since the subspace V is complete, it follows $z_i \to z \in V$, thus $-\Delta z_i \to -\Delta z \in \mathcal{M}$. Namely, $g = -\Delta z \in \mathcal{M}$. Therefore, the space $\mathcal{M}$ is closed. □

Recall the image $\mathcal{M}$ of the mapping $-\Delta$ in $H_0^1(\Omega)$. Let $\mathcal{M}^{\perp}$ be its orthogonal complement in $H_0^1(\Omega)$. Namely, for any function $v \in H_0^1(\Omega)$, there exist unique $v_{\mathcal{M}} \in \mathcal{M}$ and $v_{\perp} \in \mathcal{M}^{\perp}$ such that

$$v = v_{\mathcal{M}} + v_{\perp} \tag{2.8}$$

and

$$(\nabla v_{\mathcal{M}}, \nabla v_{\perp}) = 0 \tag{2.9}$$

In other words, $\mathcal{M} \oplus \mathcal{M}^{\perp} = H_0^1(\Omega)$. By the definition of $\mathcal{M}$, the condition (2.9) is equivalent to

$$(\nabla \Delta z, \nabla v_{\perp}) = 0, \quad \forall z \in V.$$

In the following, we will show that the space $\mathcal{M}^{\perp}$ is finite-dimensional, allowing for the determination of its basis.

Denote the ℓth side of $\partial\Omega$ by $\overline{\Gamma}_{\ell}$, where Γ_{ℓ} is open. For $\forall \phi, \psi \in H^4(\Omega)$, Green's formula gives

$$\begin{aligned}\int_{\Omega} \phi \Delta^2 \psi \, dx &- \int_{\Omega} \Delta^2 \phi \psi \, dx \\ &= \sum_{\ell} \int_{\Gamma_{\ell}} \phi \partial_{\mathbf{n}}(\Delta\psi) - \partial_{\mathbf{n}}\phi\Delta\psi + \Delta\phi\partial_{\mathbf{n}}\psi - \partial_{\mathbf{n}}(\Delta\phi)\psi \, ds,\end{aligned} \tag{2.10}$$

where $\mathbf{n}$ is the outward normal derivative.

We denote by $\mathcal{D}(\Omega)$ the space of infinitely differentiable functions with compact support in Ω. Then we can show the following result.

Lemma 2.3. *A function v belongs to $\mathcal{M}^{\perp}$ if and only if $v \in H_0^1(\Omega)$ is the solution of the following (adjoint) boundary value problem*

$$\Delta^2 v = 0 \text{ in } \Omega, \qquad v = 0, \quad \Delta v = 0 \text{ on } \partial\Omega \tag{2.11}$$

Proof. (⇒) By (2.9), it holds for $\forall v \in \mathcal{M}^{\perp}$ and $z \in V$,

$$(-\nabla\Delta z, \nabla v) = 0 \tag{2.12}$$

In particular, for $\forall z \in \mathcal{D}(\Omega) \subset V$,

$$(-\nabla\Delta z, \nabla v) = 0 = (z, \Delta^2 v), \tag{2.13}$$

which implies $\Delta^2 v = 0$ in Ω.

Define $D(\Delta^2, H^{-1}(\Omega))$ to be the maximal extension of the biharmonic operator in $H_0^1(\Omega)$:

$$D(\Delta^2, H^{-1}(\Omega)) := \{v \in H_0^1(\Omega) : \ \Delta^2 v \in H^{-1}(\Omega)\}.$$

Note that $v \in \mathcal{M}^{\perp} \subset D(\Delta^2, H^{-1}(\Omega))$. Now, suppose $v \in H^4(\Omega) \cap \mathcal{M}^{\perp}$. By Green's formula, it holds for $z \in V$,

$$\begin{aligned}(-\nabla\Delta z, \nabla v) = 0 = (z, \Delta^2 v) - \sum_{\ell} \int_{\Gamma_{\ell}} z\partial_{\mathbf{n}}(\Delta v) - \partial_{\mathbf{n}} z \Delta v \\ + \Delta z \partial_{\mathbf{n}} v \, ds.\end{aligned} \tag{2.14}$$

Then (2.13), (2.14) together with the boundary condition $z = \Delta z = 0$ on every Γ_{ℓ} yields the boundary value condition $\Delta v = 0$ on Γ_{ℓ}. Given that $H^4(\Omega)$ is dense in $D(\Delta^2, H^{-1}(\Omega))$ [26], the density argument asserts that the same boundary condition also holds for any $v \in \mathcal{M}^{\perp} \subset H_0^1(\Omega)$. Consequently, (2.11) holds.

(⇐) For $v \in H_0^1(\Omega)$ satisfying (2.11), it follows $v \in D(\Delta^2, H^{-1}(\Omega))$. Suppose $v \in H^4(\Omega) \cap D(\Delta^2, H^{-1}(\Omega))$. By (2.11) and Green's formula, (2.14) also holds for $\forall z \in V$. Since $H^4(\Omega)$ is dense in $D(\Delta^2, H^{-1}(\Omega))$, the equality $(-\nabla\Delta z, \nabla v) = 0$ also holds for $v \in D(\Delta^2, H^{-1}(\Omega))$ and implies $v \in \mathcal{M}^{\perp}$. □

One of the main goals of this section is to show that $\mathcal{M}^{\perp}$ is finite dimensional and to identify the basis of $\mathcal{M}^{\perp}$. Next, we introduce some pertinent functions in the domain Ω.

Definition 2.4. Given $R > 0$ such that $K_{\omega}^{R} \subset \Omega$. Let $N \geq 0$ be the largest integer satisfying $N < \frac{2\omega}{\pi}$ with values specified in Table 1. Additionally, let $\tau \in (0, 1)$ be a given parameter. (i) For $1 \leq i \leq N$, we define the $H^{-1}(\Omega)$ functions,

$$\xi_i(r, \theta; \tau, R) := \chi_i(r, \theta; \tau, R) + \zeta_i(r, \theta; \tau, R) \tag{2.15}$$

where

$$\chi_i(r, \theta; \tau, R) = \eta(r; \tau, R) r^{-\frac{i\pi}{\omega}} \sin\left(\frac{i\pi}{\omega}\theta\right), \tag{2.16}$$

with the cut-off function $\eta(r; \tau, R) \in C^{\infty}(\Omega)$ satisfying $\eta(r; \tau, R) = 1$ for $0 \leq r \leq \tau R$ and $\eta(r; \tau, R) = 0$ for $r > R$, and $\zeta_i \in H_0^1(\Omega)$ is obtained by solving

$$-\Delta\zeta_i = \Delta\chi_i \text{ in } \Omega, \qquad \zeta_i = 0 \text{ on } \partial\Omega \tag{2.17}$$

(ii) For $1 \leq i \leq N$, we define $\sigma_i \in H_0^1(\Omega)$ satisfying

$$-\Delta\sigma_i = \xi_i \text{ in } \Omega, \qquad \sigma_i = 0 \text{ on } \partial\Omega \tag{2.18}$$

Remark 2.5. If $N = 0$, both the function sets $\{\xi_i\}_{i=1}^{N}$ and $\{\sigma_i\}_{i=1}^{N}$ are empty. The functions $\xi_i, i = 1, \ldots, N$ defined in (2.15) are not in $H^1(\Omega)$.

TABLE 1 | The range of $\frac{\pi}{\omega}$ and the value of N for different ω in Definition 2.4.

$\boldsymbol{\omega}$	$(0, \frac{\pi}{2}]$	$(\frac{\pi}{2}, \pi)$	$(\pi, \frac{3\pi}{2}]$	$(\frac{3\pi}{2}, 2\pi)$
$\frac{\pi}{\omega}$	$(2, \infty)$	$(1, 2)$	$(\frac{2}{3}, 1)$	$(\frac{1}{2}, \frac{2}{3})$
N	0	1	2	3

Define $D(\Delta, H^{-1}(\Omega))$ to be the maximal extension of the Laplace operator in $H^{-1}(\Omega)$ [26],

$$D(\Delta, H^{-1}(\Omega)) := \{v \in H^{-1}(\Omega) : \ \Delta v \in H^{-1}(\Omega)\}.$$

For the functions ξ_i in Section 2.4, the following properties hold.

Lemma 2.6. *Given $\eta \in C^\infty(\Omega)$ in Definition2.4, the functions $\xi_i \in D(\Delta, H^{-1}(\Omega))$, $i = 1, \dots, N$, are uniquely defined and satisfy*

$$-\Delta\xi_i = 0 \text{ in } \Omega, \qquad \xi_i = 0 \text{ on } \partial\Omega \tag{2.19}$$

Moreover, ξ_i depends on the domain Ω, but not on τ and R. Namely, for any positive numbers τ_1, τ_2 and R_1, R_2, it holds

$$\xi_i(r,\theta) := \xi_i(r,\theta;\tau_1,R_1) = \xi_i(r,\theta;\tau_2,R_2) \tag{2.20}$$

Proof. For χ_i given in (2.16) with $1 \le i \le N$, it can be verified that $\chi_i \in C^\infty(\Omega \setminus K_\omega^\delta)$ for any $\delta > 0$ and $\chi_i = 0$ for $(r\cos\theta, r\sin\theta) \in \Omega \setminus K_\omega^R$. Moreover, $\Delta\chi_i = 0$ if $r < \tau R$ and $r > R$. These imply that $\Delta\chi_i \in C^\infty(\Omega) \subset L^2(\Omega)$. Given $\eta \in C^\infty(\Omega)$, the explicit function χ_i belonging to $H^{-1}(\Omega)$ in (2.16) is uniquely defined, so is $\Delta\chi_i$. In addition, $\zeta_i \in H_0^1(\Omega)$ is uniquely defined via (2.17). Therefore, ξ_i in (2.15) is uniquely defined due to the uniqueness of χ_i and ζ_i.

Taking $-\Delta$ on both side of (2.15) yields

$$-\Delta\xi_i = -\left(\Delta\chi_i + \Delta\zeta_i\right) = 0,$$

where (2.17) have been applied. In addition, $\xi_i = 0$ on $\partial\Omega$ is obtained by $\chi_i = 0$ and $\zeta_i = 0$ on $\partial\Omega$.

Next, we prove (2.20). By taking $\delta \in (0, \min\{\tau_1 R_1, \tau_2 R_2\})$, it follows $K_\omega^\delta \subset K_\omega^{\tau_1 R_1} \cap K_\omega^{\tau_2 R_2} \subset \Omega$. By (2.16), we have

$$\chi_i(r,\theta;\tau_1,R_1) - \chi_i(r,\theta;\tau_2,R_2) = 0, \quad (r\cos\theta, r\sin\theta) \in K_\omega^\delta.$$

Recall that $\chi_i(r,\theta;\tau_j,R_j) \in C^\infty(\Omega \setminus K_\omega^\delta)$, $j = 1, 2$. Then it follows

$$\chi_i(r,\theta;\tau_1,R_1) - \chi_i(r,\theta;\tau_2,R_2) \in C^\infty(\Omega).$$

Since $\zeta_i(r,\theta;\tau_j,R_j) \in H_0^1(\Omega)$, $j = 1, 2$, we have

$$\begin{aligned} \tilde{\xi}_i :&= \xi_i(r,\theta;\tau_1,R_1) - \xi_i(r,\theta;\tau_2,R_2) \\ &= \zeta_i(r,\theta;\tau_1,R_1) - \zeta_i(r,\theta;\tau_2,R_2) \\ &\quad + \left(\chi_i(r,\theta;\tau_1,R_1) - \chi_i(r,\theta;\tau_2,R_2)\right) \in H_0^1(\Omega). \end{aligned}$$

Meanwhile, from (2.19), we have

$$\begin{aligned} \Delta\tilde{\xi}_i &= \Delta\xi_i(r,\theta;\tau_1,R_1) - \Delta\xi_i(r,\theta;\tau_2,R_2) = 0 \text{ in } \Omega, \\ \tilde{\xi}_i &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{2.21}$$

By applying the Lax-Milgram Theorem to (2.21), it is established that $\tilde{\xi}_i = 0$, indicating the validity of (2.20). □

Remark 2.7. Lemma 2.6 implies that $\xi_i(r,\theta;\tau,R)$ in Section 2.4 can be replaced by $\xi_i(r,\theta)$. Moreover, the $H^{-1}(\Omega)$ functions $\xi_i(r,\theta) \not\equiv 0$, because otherwise we have $\chi_i = -\zeta_i \in H_0^1(\Omega)$, which contradicts the fact that $\chi_i \notin H^1(\Omega)$.

Subsequently, for the functions σ_i in $H_0^1(\Omega)$ defined in Section 2.4, the following property is satisfied.

Lemma 2.8. *The functions $\sigma_i \in D(\Delta^2, H^{-1}(\Omega))$, $i = 1, \dots, N$, in Definition 2.4 are uniquely defined and satisfy*

$$\Delta^2\sigma_i = 0 \text{ in } \Omega, \qquad \sigma_i = 0, \quad \Delta\sigma_i = 0 \text{ on } \partial\Omega \tag{2.22}$$

Proof. Note that σ_i is obtained through the Poisson problem (2.18) with ξ_i as the source term. From Lemma 2.6, ξ_i is uniquely defined, which yields the uniqueness σ_i. Applying $-\Delta$ to (2.18) in conjunction with (2.19) yields (2.22). □

For both functions ξ_i and σ_i, we have the following results.

Lemma 2.9. (*a*) *The functions $\xi_i(= \Delta\sigma_i)$, $i = 1, 2, \dots, N$, are linearly independent*. (*b*) *The functions $\sigma_i, \nabla\sigma_i$, $i = 1, 2, \dots, N$, are also linearly independent, respectively*.

Proof. (a) $\xi_i(r,\theta) \not\equiv 0$, because otherwise we have $\chi_i = -\zeta_i \in H_0^1(\Omega)$, which contradicts the fact that $\chi_i \notin H_0^1(\Omega)$. We assume that

$$\sum_{i=1}^{N} C_i\xi_i = 0 \tag{2.23}$$

where C_i, $i = 1, 2, \dots, N$ are some constants. Plugging (2.15) into (2.23) gives

$$\sum_{i=1}^{N} C_i\chi_i = -\sum_{i=1}^{N} C_i\zeta_i \in H_0^1(\Omega).$$

Note that $\chi_i \notin H_0^1(\Omega)$, $i = 1, 2, \dots, N$. Therefore, it holds

$$\sum_{i=1}^{N} C_i\chi_i = 0 \tag{2.24}$$

Multiplying (2.24) by $r^{-\frac{\pi}{\omega}}$, we have $C_N r^{-\frac{\pi}{\omega}}\chi_N = -\sum_{i=1}^{N-1} C_i r^{-\frac{\pi}{\omega}}\chi_i \in H^{-1}(\Omega)$, which contradicts the fact that $C_N r^{-\frac{\pi}{\omega}}\chi_N \notin H^{-1}(\Omega)$. Thus, it follows $C_N = 0$.For $i = 2, \dots, N$, multiplying (2.24) by $r^{-\frac{i\pi}{\omega}}$, the same argument yields $C_{N+1-i} = 0$.Thus, ξ_i, $i = 1, 2, \dots, N$, are linearly independent.

(b) We assume $\sum_{i=1}^{N} C_i'\sigma_i = 0$ for some constants C_i' and apply $-\Delta$ to both sides of the equation, it follows

$$\sum_{i=1}^{N} C_i'\xi_i = 0.$$

By (a), we have $C_i' = 0$, $i = 1, \dots, N$, which implies σ_i, $i = 1, 2, \dots, N$, are linearly independent. The linear independence of $\nabla\sigma_i$ can be proved similarly. □

Corollary 2.10. *The space* $\mathrm{span}\{\sigma_i, i = 1, \dots, N\} \subset \mathcal{M}^\perp$, *and the dimension of $\mathcal{M}$ satisfies* $\dim(\mathcal{M}^\perp) \ge N$.

The proof follows from Section 2.3, Lemma 2.8, and Section 2.9.

Lemma 2.11. *For any function $v \in H_0^1(\Omega)$, it holds*

$$\langle v, \xi_i \rangle = (\nabla v, \nabla\sigma_i), \quad \forall i = 1, \dots, N \tag{2.25}$$

where ξ_i andσ_i are given in Section 2.4. In particular, if $v \in \mathcal{M}$, it holds

$$\langle v, \xi_i \rangle = (\nabla v, \nabla \sigma_i) = 0, \quad \forall i = 1, \dots, N \tag{2.26}$$

Proof. Multiplying (2.18) by $v \in H_0^1(\Omega)$ and applying Green's formula yield (2.25). Since $\sigma_i \in \mathcal{M}^\perp$, (2.26) follows from (2.9). □

To determine the dimension of $\mathcal{M}^\perp$, we let λ_i^2 be the eigenvalues to the following one dimensional problem

$$-\partial_{\theta\theta}\phi_i = \lambda_i^2 \phi_i \quad \text{in } (0, \omega), \qquad \phi(0) = \phi(\omega) = 0 \tag{2.27}$$

For $i \ge 1$, it is clear that when $\lambda_i > 0$,

$$\lambda_i = \frac{i\pi}{\omega}, \quad \phi_i = \sqrt{\frac{2}{\omega}} \sin\left(\frac{i\pi}{\omega}\theta\right) \tag{2.28}$$

In addition, we also recall the following result for the Poisson problem (2.6).

Lemma 2.12. *Assume that $g \in H^1(\Omega)$, and $\lambda_i = \frac{i\pi}{\omega}$, $1 \le i \le N$ for N given in Table 1, is not an integer, namely $\omega \ne \frac{\pi}{2}$ and $\omega \ne \frac{3\pi}{2}$. Then the solution z of the Poisson problem(2.6) from the space $H^{1+\alpha}(\Omega) \cap H_0^1(\Omega)$ for $\alpha < \frac{\pi}{\omega}$ possesses the asymptotic representation in the neighborhood of Q,*

$$z(x) = \tilde{z}(x) + \eta(r) \sum_{i=1}^{N} d_i (i\pi)^{-\frac{1}{2}} r^{\frac{i\pi}{\omega}} \sin\left(\frac{i\pi\theta}{\omega}\right) \tag{2.29}$$

where $\tilde{z}(x) \in H^3(\Omega) \cap H_0^1(\Omega)$ and the coefficients d_i are defined by

$$d_i = \langle g, \xi_i \rangle, \quad i = 1, \dots, m \tag{2.30}$$

Moreover, it follows that

$$\|\tilde{z}\|_{H^3(\Omega)} + \sum_{i=1}^{N} |d_i| \le C \|g\|_{H^1(\Omega)} \tag{2.31}$$

Proof. The proof can be found in Theorem 3.4 in [25] and Section 2.7 in [21]. □

Based on Section 2.12, we can identify the dimension of $\mathcal{M}^\perp$ as follows.

Lemma 2.13. *Under the condition in Section 2.12. The dimension of $\mathcal{M}^\perp$ is equal to the cardinality of the set $\{\lambda_i : 0 < \lambda_i < 2\}$, namely*

$$\dim(\mathcal{M}^\perp) = \text{card}\,\{\lambda_i : 0 < \lambda_i < 2\} = N,$$

where the condition $0 < \lambda_i < 2$ corresponds to $1 \le i \le N$.

Proof. For $\forall v \in \mathcal{M}^\perp$, by (2.9) it holds

$$(\nabla g, \nabla v) = 0, \quad \forall g \in \mathcal{M} \tag{2.32}$$

For the Poisson problem (2.6) with $g \in \mathcal{M} \subset H_0^1(\Omega)$, Section 2.2 implies that its solution $z \in V \subset H^3(\Omega)$. By Section 2.12, $z \in H^3(\Omega)$ is equivalent to the fact that for $\lambda_i \in (0, 2)$, the coefficients

$$d_i = \langle g, \xi_i \rangle = (\nabla g, \nabla \sigma_i) = 0 \tag{2.33}$$

where we have used (2.25) in the second equality. If λ_i is not an integer, $\lambda_i \in (0, 2)$ corresponds to the integer $i \in [1, N]$. (2.32) and (2.33) imply that $\mathcal{M}^\perp \subset \text{span}\{\sigma_i,\ i = 1, \dots, N\}$, which together with Section 2.10 gives the conclusion. □

The cases of $\omega = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ are not covered by Section 2.12. To address this limitation, we introduce the following additional result.

Remark 2.14. The asymptotic representation of the solution z to problem (2.6) typically involves two types of singular functions depending on $\lambda_i = i\pi/\omega$:

$$S_i = (i\pi)^{-\frac{1}{2}} r^{\frac{i\pi}{\omega}} \sin\left(\frac{i\pi\theta}{\omega}\right) \quad \text{when } \lambda_i \text{ is not an integer} \tag{2.34a}$$

$$S_i = r^{\frac{i\pi}{\omega}} \left(\ln r \sin\left(\frac{i\pi\theta}{\omega}\right) + \theta \cos\left(\frac{i\pi\theta}{\omega}\right) \right) \quad \text{otherwise} \tag{2.34b}$$

Specially, the coefficient of the term in (2.34b) depends locally on the restriction of the data g to any neighborhood of the corner [28]. If $g \in H_0^1(\Omega)$, the solution of problem (2.6) has the expansion [28]

$$z - \sum_{0<\lambda_i<2} d_i S_i \in H^3(\Omega) \tag{2.35}$$

where d_i is given by (2.30). In other words, when the source term $g \in H_0^1(\Omega)$, the singular function S_i in (2.34b) with $\lambda_i = i\pi/\omega = 2$ vanishes in the asymptotic representation of z.

Corollary 2.15. *The dimension of $\mathcal{M}^\perp$ satisfies $\dim(\mathcal{M}^\perp) = N$. Moreover,*

$$\text{span}\{\sigma_i,\ i = 1, \dots, N\} = \mathcal{M}^\perp.$$

Proof. The proof follows from Sections 2.9, and 2.10, Lemma 2.13, and Section 2.14. □

For $\forall v \in H_0^1(\Omega)$, Section 2.15 and (2.9) imply that $(\nabla v_{\mathcal{M}}, \nabla \sigma_i) = 0$, $1 \le i \le N$ and that there exists a unique decomposition,

$$v = v_{\mathcal{M}} + \sum_{i=1}^{N} c_i \sigma_i \tag{2.36}$$

where $v_{\mathcal{M}} \in \mathcal{M}$ and the coefficients c_i are uniquely determined by the linear system,

$$\sum_{i=1}^{N} c_i (\nabla \sigma_i, \nabla \sigma_j) = (\nabla v, \nabla \sigma_j), \quad j = 1, \dots, N \tag{2.37}$$

By Section 2.11, it holds that for $\forall \phi \in H_0^1(\Omega)$,

$$(\nabla \sigma_j, \nabla \phi) = \langle \xi_j, \phi \rangle, \quad j = 1, \dots, N \tag{2.38}$$

Therefore, the linear system (2.39) is equivalent to the following linear system

$$\sum_{i=1}^{N} c_i \langle \sigma_i, \xi_j \rangle = \langle v, \xi_j \rangle, \quad j = 1, \dots, N. \tag{2.39}$$

Lemma 2.16. *The linear system (2.39) or (2.37) admits a unique solution c_i, $i = 1, \dots, N$.*

Proof. Since (2.39) and (2.37) are equivalent, we only need to consider (2.37), which is a finite-dimensional linear system. The existence of the solution is equivalent to the uniqueness. Let $\overline{c}_i$ be the difference between two possible solutions; it follows

$$\left(\sum_{i=1}^{N}\overline{c}_i\nabla\sigma_i, \nabla\sigma_j\right) = \sum_{i=1}^{N}\overline{c}_i(\nabla\sigma_i, \nabla\sigma_j) = 0, \quad j = 1, \ldots, N.$$

A linear combination in terms of $\nabla\sigma_j$ gives

$$\left(\sum_{i=1}^{N}\overline{c}_i\nabla\sigma_i, \sum_{j=1}^{N}\overline{c}_j\nabla\sigma_j\right) = 0,$$

which means $\left\|\sum_{i=1}^{N}\overline{c}_i\nabla\sigma_i\right\| = 0$, thus we have

$$\sum_{i=1}^{N}\overline{c}_i\nabla\sigma_i = 0.$$

Section 2.9 indicates $\overline{c}_i = 0$, $i = 1, \ldots, N$. Thus, the conclusion holds. □

2.4 | The Modified Mixed Formulation

Based on the discussion above, we propose a modified mixed formulation for (1.1),

$$\begin{cases} -\Delta w = f & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega; \end{cases} \quad \begin{cases} -\Delta v = w & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega; \end{cases}$$
$$\begin{cases} -\Delta \tilde{u} = v - \sum_{i=1}^{N} c_i\sigma_i & \text{in } \Omega, \\ \tilde{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.40}$$

where σ_i are given in (2.18) and c_i are given by (2.37).

The modified mixed weak formulation for (2.40) is to find $w, v, \tilde{u} \in H_0^1(\Omega)$ such that

$$A(w, \phi) = (f, \phi) \tag{2.41a}$$

$$A(v, \phi) = (w, \psi) \tag{2.41b}$$

$$A(\tilde{u}, \tau) = \left(v - \sum_{i=1}^{N} c_i\sigma_i, \tau\right) \tag{2.41c}$$

for any $\phi, \psi, \tau \in H_0^1(\Omega)$.

Next, we show that $\tilde{u}$ is the weak solution to the variational formulation (2.2).

Theorem 2.17. *Given $f \in H^{-1}(\Omega)$, let $\tilde{u}$ be the solution of the modified mixed weak formulation* (2.41). *Then$\tilde{u}$ is equivalent to the solution of the weak formulation (2.2), namely, $u = \tilde{u}$ in V, and vice versa.*

Proof. Note that $v, \sigma_i \in H_0^1(\Omega)$. Thus $v - \sum_{i=1}^{N} c_i\sigma_i \in H_0^1(\Omega)$. By (2.39), it holds $d_j = \langle v - \sum_{i=1}^{N} c_i\sigma_i, \xi_j\rangle = 0$, $j = 1, \ldots, N$. Therefore, by applying Lemma 2.12 and Section 2.14 to the last Poisson equation in (2.40), it follows $\tilde{u} \in H^3(\Omega) \cap H_0^1(\Omega)$. Since $\Delta\tilde{u}|_{\partial\Omega} = -(v - \sum_{i=1}^{N} c_i\sigma_i)|_{\partial\Omega} = 0$, it follows $\tilde{u} \in V$.

On the other hand,

$$-\Delta^3\tilde{u} = \Delta^2 v - \sum_{i=1}^{N} c_i\Delta^2\sigma_i = -\Delta(-\Delta v) = -\Delta w = f,$$

where we have used the result (2.22). Thus, we have $\tilde{u} \in V$ satisfying (2.2). Finally, by the uniqueness of the solution of (2.2) in V, the conclusion holds. □

Therefore, by Theorem 2.17, the solution u of the sixth-order problem (1.1) satisfies

$$\begin{cases} -\Delta w = f & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega; \end{cases} \quad \begin{cases} -\Delta v = w & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega; \end{cases}$$
$$\begin{cases} -\Delta u = v - \sum_{i=1}^{N} c_i\sigma_i & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.42}$$

The corresponding weak formulation is to find $w, v, u \in H_0^1(\Omega)$ such that for any $\phi, \psi, \tau \in H_0^1(\Omega)$,

$$A(w, \phi) = (f, \phi) \tag{2.43a}$$

$$A(v, \psi) = (w, \psi) \tag{2.43b}$$

$$A(u, \tau) = \left(v - \sum_{i=1}^{N} c_i\sigma_i, \tau\right) \tag{2.43c}$$

where c_i, $i = 1, \ldots, N$, are given in (2.37).

Remark 2.18. For the following cases, the modified mixed formulation (2.42) is identical to the direct mixed formulation (2.4): (i) $N = 0$, which happens if $\omega \le \frac{\pi}{2}$ as shown in Table 1; (ii) the boundary of domain Ω is sufficiently smooth; (iii) $c_i = 0$, $i = 1, \ldots, N$ in (2.39) or (2.37), which is possible for some source term f such that the solution $v \in \mathcal{M}$ in (2.42).

Lemma 2.19. *The mapping $v \to v_{\mathcal{M}}$ in (2.36) defines a norm non-increasing mapping $H_0^1(\Omega) \to \mathcal{M}$ in the sense*

$$\|\nabla v_{\mathcal{M}}\| \le \|\nabla v\|.$$

Proof. Multiplying (2.36) by $-\Delta v_{\mathcal{M}}$, integrating over the domain Ω, and applying Green's Theorem give

$$(\nabla v, \nabla v_{\mathcal{M}}) = (\nabla v_{\mathcal{M}}, \nabla v_{\mathcal{M}}) + \left(\sum_{i=1}^{N} c_i\nabla\sigma_i, \nabla v_{\mathcal{M}}\right) \tag{2.44}$$

Note that

$$(\nabla v_{\mathcal{M}}, \nabla\sigma_j) = \left(\nabla\left(v - \sum_{i=1}^{N} c_i\sigma_i\right), \nabla\sigma_j\right)$$
$$= (\nabla v, \nabla\sigma_j) - \sum_{i=1}^{N} c_i\left(\nabla\sigma_i, \nabla\sigma_j\right) = 0, \quad j = 1, \ldots, N,$$

where we have used (2.37) in the last equality. For the last term in (2.44), it follows

$$\left(\sum_{i=1}^{N} c_i \nabla \sigma_i, \nabla v_{\mathcal{M}}\right) = 0.$$

Then, applying Hölder's inequality to (2.44), it follows

$$\|\nabla v_{\mathcal{M}}\|^2 = (\nabla v, \nabla v_{\mathcal{M}}) \le \|\nabla v_{\mathcal{M}}\| \|\nabla v\|,$$

which gives the conclusion. □

In addition, we have the following regularity result.

Theorem 2.20. *Given $f \in H^{-1}(\Omega)$, for w, u, v in(2.42), it follows*

$$\|w\|_{H^1(\Omega)} \le C\|f\|_{H^{-1}(\Omega)} \tag{2.45a}$$

$$\|v\|_{H^1(\Omega)} \le C\|f\|_{H^{-1}(\Omega)} \tag{2.45b}$$

$$\|u\|_{H^3(\Omega)} \le C\|f\|_{H^{-1}(\Omega)} \tag{2.45c}$$

Proof. The estimate (2.45a) is a direct consequence of the fact that the Laplace operator is an isomorphism between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. In a similar fashion,

$$\|v\|_{H^1(\Omega)} \le \|w\|_{H^{-1}(\Omega)} \le C\|w\|_{H^1(\Omega)} \le C\|f\|_{H^{-1}(\Omega)},$$

which gives the estimate (2.45b). By Theorem 2.17, it follows $u \in V$. Moreover, (2.31) gives

$$\begin{aligned}\|u\|_{H^3} &\le C\left\| v - \sum_{i=1}^{N} c_i \sigma_i \right\|_{H^1(\Omega)} \\ &= C\|v_{\mathcal{M}}\|_{H^1(\Omega)} \le C\|v\|_{H^1(\Omega)} \le C\|f\|_{H^{-1}(\Omega)},\end{aligned} \tag{2.46}$$

where we have used Lemma 2.19 and Poincaré inequality. □

3 | The Finite Element Method

In this section, we introduce a linear C^0 finite element method for solving the sixth-order problem (1.1). Subsequently, we conduct a finite element error analysis.

3.1 | The Finite Element Algorithm

Let $\mathcal{T}_n$ denote a triangulation of Ω consisting of shape-regular triangles, and let $S_n \subset H_0^1(\Omega)$ be the C^0 Lagrange linear finite element space associated with $\mathcal{T}_n$. Then we proceed to propose the finite element algorithm.

Algorithm 3.1. *We define the finite element solution of the sixth-order problem (1.1) by employing the decoupling presented in* (2.43) *as follows*:

- *Step* 1. *Find the finite element solution $w_n \in S_n$ of the Poisson equation*
$$A(w_n, \phi) = (f, \phi), \qquad \forall \phi \in S_n \tag{3.1}$$
- *Step* 2. *Find the finite element solution $v_n \in S_n$ of the Poisson equation*
$$A(v_n, \psi) = (w_n, \psi), \qquad \forall \psi \in S_n \tag{3.2}$$
- *Step* 3. *With χ_i, $i = 1, \dots, N$ defined in (2.16), we compute the finite element solution $\zeta_{i,n} \in S_n$ of the Poisson equation*
$$A(\zeta_{i,n}, \phi) = (\Delta \chi_i, \phi), \qquad \forall \phi \in S_n \tag{3.3}$$
and set $\xi_{i,n} = \zeta_{i,n} + \chi_i$.
- *Step* 4. *Find the finite element solution $\sigma_{i,n} \in S_n$, $i = 1, \dots, N$ of the Poisson equation*
$$A(\sigma_{i,n}, \phi) = (\xi_{i,n}, \phi), \qquad \forall \phi \in S_n \tag{3.4}$$
- *Step* 5. *Find the coefficient $c_{i,n} \in \mathbb{R}$ by solving the linear system*
$$\sum_{i=1}^{N} c_{i,n} \langle \sigma_{i,n}, \xi_{j,n} \rangle = \langle v_n, \xi_{j,n} \rangle, \quad j = 1, \dots, N \tag{3.5}$$
- *Step* 6. *Find the finite element solution $u_n \in S_n$ of the Poisson equation*
$$A(u_n, \tau) = \left(v_n - \sum_{i=1}^{N} c_{i,n} \sigma_{i,n}, \tau\right), \qquad \forall \tau \in S_n \tag{3.6}$$

Remark 3.2. According to (3.3), $\zeta_{i,n} \in S_n$, while $\xi_{i,n} \in H^{-1}(\Omega)$ but $\xi_{i,n} \notin S_n$. In addition, the finite element approximations in Algorithm 3.1 are well defined based on the Lax–Milgram Theorem.

For the functions in (3.1), the following results hold.

Lemma 3.3.

a. *The $H^{-1}(\Omega)$ functions $\xi_{i,n}$, $i = 1, 2, \dots, N$, are linearly independent.*
b. *The functions $\sigma_{i,n}, \nabla \sigma_{i,n}$, $i = 1, 2, \dots, N$, are also linearly independent, respectively.*

Proof.

a. The proof is similar to the proof of Theorem 2.9a.
b. We assume that $\sum_{i=1}^{N} C_i' \sigma_{i,n} = 0$ for some constants C_i'. The combination of (3.4) gives
$$\left(\sum_{i=1}^{N} C_i' \xi_{i,n}, \phi\right) = A\left(\sum_{i=1}^{N} C_i' \sigma_{i,n}, \phi\right) = 0.$$
By (a), we have $C_i' = 0$, $i = 1, \dots, N$, which implies $\sigma_{i,n}$, $i = 1, 2, \dots, N$, are linearly independent. The linear independence of $\nabla \sigma_{i,n}$ can be proved similarly. □

3.2 | Optimal Error Estimates on Quasi-Uniform Meshes

Suppose that the mesh $\mathcal{T}_n$ consists of quasi-uniform triangles with size h. Recall the interpolation error estimates [10] on $\mathcal{T}_n$ for any $z \in H^{1+s}(\Omega)$, $s > 0$,

$$\|z - z_I\|_{H^m(\Omega)} \le Ch^{\min\{s+1,2\}-m}\|z\|_{H^{\min\{s+1,2\}}(\Omega)} \tag{3.7}$$

where $m = 0, 1$ and $z_I \in S_n$ represents the nodal interpolation of z. Let $z_n \in S_n$ be the finite element solution of the Poisson Equation (2.6) in the polygonal domain, if $z \in H^{1+s}(\Omega)$, $s > 0$, the standard error estimate [10, 29] yields

$$\begin{aligned} \|z - z_n\|_{H^1(\Omega)} &\le Ch^{\min\{s,1\}}\|z\|_{H^{1+\min\{s,1\}}(\Omega)}, \\ \|z - z_n\| &\le Ch^{2\min\{s,1\}}\|z\|_{H^{1+\min\{s,1\}}(\Omega)}. \end{aligned} \tag{3.8}$$

Given $g \in L^2(\Omega)$ in (2.6), it is well known that the solution $z \in H^{1+\alpha}(\Omega)$ with $\alpha < \frac{\pi}{\omega}$ (see e.g., [21, 28, 29]). Note that $f, \Delta\chi_i \in L^2(\Omega)$ in Poisson Equations (2.17) and (2.42), so it follows $w, \zeta_i \in H^{1+\alpha}(\Omega)$. Note that $\xi_i \in H^{-1}(\Omega)$, but Step 3 in Algorithm 3.1 indicates $\xi_i - \xi_{i,n} = \zeta_i - \zeta_{i,n}$. Therefore, we have the following error estimates:

Lemma 3.4. *Given w_n and $\xi_{i,n}$ in Algorithm 3.1, it follows*

$$\|w - w_n\|_{H^1(\Omega)} \le Ch^{\min\{\alpha,1\}}\|w\|_{H^{1+\min\{\alpha,1\}}(\Omega)} \tag{3.9a}$$

$$\|w - w_n\| \le Ch^{2\min\{\alpha,1\}}\|w\|_{H^{1+\min\{\alpha,1\}}(\Omega)} \tag{3.9b}$$

$$\|\xi_i - \xi_{i,n}\|_{H^1(\Omega)} \le Ch^{\min\{\alpha,1\}}\|\zeta_i\|_{H^{1+\min\{\alpha,1\}}(\Omega)} \tag{3.9c}$$

$$\|\xi_i - \xi_{i,n}\|_{H^{-1}(\Omega)} \le C\|\xi_i - \xi_{i,n}\| \le Ch^{2\min\{\alpha,1\}}\|\zeta_i\|_{H^{1+\min\{\alpha,1\}}(\Omega)} \tag{3.9d}$$

Note that the basis $\{\sigma_i\}_{i=1}^N$ given in Section 2.4 is not orthogonal if $\omega > \pi$. For analysis convenience, we can apply Schmidt orthogonalization to obtain an orthogonal basis $\{\tilde\sigma_i\}_{i=1}^N$,

$$\begin{aligned} \tilde\sigma_1 &= \sigma_1, \\ \tilde\sigma_2 &= \sigma_2 - \frac{(\nabla\sigma_2, \nabla\tilde\sigma_1)}{\|\nabla\tilde\sigma_1\|^2}\tilde\sigma_1, \\ \tilde\sigma_3 &= \sigma_3 - \frac{(\nabla\sigma_3, \nabla\tilde\sigma_1)}{\|\nabla\tilde\sigma_1\|^2}\tilde\sigma_1 - \frac{(\nabla\sigma_3, \nabla\tilde\sigma_2)}{\|\nabla\tilde\sigma_2\|^2}\tilde\sigma_2. \end{aligned} \tag{3.10}$$

Namely, $(\nabla\tilde\sigma_i, \nabla\tilde\sigma_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta function. Furthermore, we denote $\{\tilde\xi_i\}_{i=1}^N$ by

$$\begin{aligned} \tilde\xi_1 &= \xi_1, \\ \tilde\xi_2 &= \xi_2 - \frac{(\nabla\sigma_2, \nabla\tilde\sigma_1)}{\|\nabla\tilde\sigma_1\|^2}\tilde\xi_1, \\ \tilde\xi_3 &= \xi_3 - \frac{(\nabla\sigma_3, \nabla\tilde\sigma_1)}{\|\nabla\tilde\sigma_1\|^2}\tilde\xi_1 - \frac{(\nabla\sigma_3, \nabla\tilde\sigma_2)}{\|\nabla\tilde\sigma_2\|^2}\tilde\xi_2. \end{aligned} \tag{3.11}$$

It can be verified that

$$-\Delta\tilde\sigma_i = \tilde\xi_i \text{ in } \Omega, \qquad \tilde\sigma_i = 0 \text{ on } \partial\Omega \tag{3.12}$$

and its weak formulation is to find $\tilde\sigma_i \in H_0^1(\Omega)$ such that $\forall\phi \in H_0^1(\Omega)$,

$$A(\tilde\sigma_i, \phi) = \langle\tilde\xi_i, \phi\rangle \tag{3.13}$$

With the new basis $\{\tilde\sigma_i\}_{i=1}^N$, the third Poisson problem in (2.42) can be equivalently written as

$$\begin{cases} -\Delta u = v - \sum_{i=1}^N \tilde c_i\tilde\sigma_i & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.14}$$

where the coefficients

$$\tilde c_i = \frac{\langle v, \tilde\xi_i\rangle}{\langle\tilde\sigma_i, \tilde\xi_i\rangle}, \quad i = 1, \ldots, N \tag{3.15}$$

or equivalently,

$$\tilde c_i = \frac{(\nabla v, \nabla\tilde\sigma_i)}{\|\nabla\tilde\sigma_i\|^2}, \quad i = 1, \ldots, N \tag{3.16}$$

Correspondingly, the weak formulation (2.43c) becomes

$$A(u, \tau) = \left(v - \sum_{i=1}^N \tilde c_i\tilde\sigma_i, \tau\right) \tag{3.17}$$

Similarly, we apply the Schmidt orthogonalization to obtain an orthogonal basis $\{\tilde\sigma_{i,n}\}_{i=1}^N$,

$$\begin{aligned} \tilde\sigma_{1,n} &= \sigma_{1,n}, \\ \tilde\sigma_{2,n} &= \sigma_{2,n} - \frac{(\nabla\sigma_{2,n}, \nabla\tilde\sigma_{1,n})}{\|\nabla\tilde\sigma_{1,n}\|^2}\tilde\sigma_{1,n}, \\ \tilde\sigma_{3,n} &= \sigma_{3,n} - \frac{(\nabla\sigma_{3,n}, \nabla\tilde\sigma_{1,n})}{\|\nabla\tilde\sigma_{1,n}\|^2}\tilde\sigma_{1,n} - \frac{(\nabla\sigma_{3,n}, \nabla\tilde\sigma_{2,n})}{\|\nabla\tilde\sigma_{2,n}\|^2}\tilde\sigma_{2,n}. \end{aligned} \tag{3.18}$$

Namely, $(\nabla\tilde\sigma_{i,n}, \nabla\tilde\sigma_{j,n}) = \delta_{ij}$, where δ_{ij} is the Kronecker delta function. Similarly, we take $\{\tilde\xi_i\}_{i=1}^N$,

$$\begin{aligned} \tilde\xi_{1,n} &= \xi_{1,n}, \\ \tilde\xi_{2,n} &= \xi_{2,n} - \frac{(\nabla\sigma_{2,n}, \nabla\tilde\sigma_{1,n})}{\|\nabla\tilde\sigma_{1,n}\|^2}\tilde\xi_{1,n}, \\ \tilde\xi_{3,n} &= \xi_{3,n} - \frac{(\nabla\sigma_{3,n}, \nabla\tilde\sigma_{1,n})}{\|\nabla\tilde\sigma_{1,n}\|^2}\tilde\xi_{1,n} - \frac{(\nabla\sigma_{3,n}, \nabla\tilde\sigma_{2,n})}{\|\nabla\tilde\sigma_{2,n}\|^2}\tilde\xi_{2,n}. \end{aligned} \tag{3.19}$$

For the orthogonal basis $\{\tilde\sigma_{i,n}\}_{i=1}^N$,

$$A(\tilde\sigma_{i,n}, \phi) = \langle\tilde\xi_{i,n}, \phi\rangle \tag{3.20}$$

and the last two steps of Algorithm 3.1 can be modified as

- Step 5. Find the coefficient $\tilde c_{i,n} \in \mathbb{R}$,

$$\tilde c_{i,n} = \frac{\langle v_n, \tilde\xi_{i,n}\rangle}{\langle\tilde\sigma_{i,n}, \tilde\xi_{i,n}\rangle}, \quad i = 1, \ldots, N \tag{3.21}$$

- Step 6. Find the finite element solution $u_n \in S_n$ of the Poisson equation

$$A(u_n, \tau) = \left(v_n - \sum_{i=1}^N \tilde c_{i,n}\tilde\sigma_{i,n}, \tau\right), \qquad \forall\tau \in S_n \tag{3.22}$$

To show the error estimates, we prepare the following results.

Lemma 3.5. *(i) Assume that $0 \le s \le 1$. Then for $\phi \in H_0^s(\Omega) \subset H_0^1(\Omega)$ it follows that $r^{-s}\phi \in L^2(\Omega)$ and*

$$\|r^{-s}\phi\| \le C\|\phi\|_{H^s(\Omega)} \le C\|\phi\|_{H^1(\Omega)} \tag{3.23}$$

(ii) If $\gamma \in [0,1)$, $s' \le 1+\gamma$, and $\phi \in H_0^{s'-\gamma}(\Omega) \subset H_0^1(\Omega)$, then we have $r^{-s'}\phi \in H^{-\gamma}(\Omega)$ and

$$\|r^{-s'}\phi\|_{H^{-\gamma}(\Omega)} \leq C\|r^{-s'+\gamma}\phi\| \tag{3.24}$$

Proof. (i) The estimate (3.23) follows from ([21], Theorem 1.2.15).

(ii) Since $s' - \gamma \leq 1$, then we have $r^{-s'+\gamma}\phi \in L^2(\Omega)$ by (i) and it holds $\|r^{-s'+\gamma}\phi\| \leq C\|\phi\|_{H^{s'-\gamma}(\Omega)}$. For (3.24), we have

$$\|r^{-s'}\phi\|_{H^{-\gamma}(\Omega)} := \sup_{\psi\in H_0^\gamma(\Omega)} \frac{\langle r^{-s'}\phi, \psi\rangle}{\|\psi\|_{H^\gamma(\Omega)}} = \sup_{\psi\in H_0^\gamma(\Omega)} \frac{\langle r^{-s'+\gamma}\phi, r^{-\gamma}\psi\rangle}{\|\psi\|_{H^\gamma(\Omega)}}$$
$$\leq \sup_{\psi\in H_0^\gamma(\Omega)} \frac{\|r^{-s'+\gamma}\phi\|\|r^{-\gamma}\psi\|}{\|\psi\|_{H^\gamma(\Omega)}} \leq C\|r^{-s'+\gamma}\phi\|,$$

where (3.23) is used for ψ in the last inequality. □

Next, we introduce some regularity results for a general Poisson problem (2.6).

Lemma 3.6. *For $g \in H^{\min\{\alpha-1,s\}}(\Omega)$ for any $s \in (-1, 0]$ and $\alpha \in (\frac{1}{2}, \frac{\pi}{\omega})$, then (2.6) admits a unique solution $z \in H^{\min\{\alpha+1,s+2\}}(\Omega)$ and it holds*

$$\|z\|_{H^{\min\{\alpha+1,s+2\}}(\Omega)} \leq C\|g\|_{H^{\min\{\alpha-1,s\}}(\Omega)} \tag{3.25}$$

Proof. The proof follows from ([25], Theorem 3.1). □

Lemma 3.7. *For $\beta_i \in (-1, 1 - \frac{i\pi}{\omega})$, $i = 1, \ldots, N$, and $\alpha \in (\frac{1}{2}\lfloor\frac{2\pi}{\omega}\rfloor, \frac{\pi}{\omega})$ with $\omega \in (\frac{\pi}{2}, 2\pi)$, if $\phi \in H_0^1(\Omega)$ and $g = r^{2\min\{\alpha-1,\beta_i\}}\phi$, then (2.6) admits a unique solution $z \in H^{\min\{2+\beta_i,1+\alpha\}}(\Omega)$ and holds the estimates*

$$\|z\|_{H^{\min\{2+\beta_i,1+\alpha\}}(\Omega)} \leq C\|r^{\min\{\alpha-1,\beta_i\}}\phi\| \tag{3.26}$$

Here, $\lfloor\cdot\rfloor$ represents the floor function.

Proof. (1) If $\omega \in \left(\frac{\pi}{2}, \pi\right)$, namely $\frac{\pi}{\omega} \in (1, 2)$, then $-1 < \beta_N = \beta_1 < 1 - \frac{\pi}{\omega} < 0$ and $\alpha - 1 > \frac{1}{2}\lfloor\frac{2\pi}{\omega}\rfloor - 1 \geq 0$. Consequently, it holds $\min\{\alpha - 1, \beta_i\} = \beta_i \in (-1, 0)$.

(2) If $\omega \in (\pi, 2\pi)$, it follows $\alpha \in \left(\frac{1}{2}, \frac{\pi}{\omega}\right) \subset \left(\frac{1}{2}, 1\right)$, implying $\alpha - 1 \in \left(-\frac{1}{2}, 0\right)$. This, together with the assumption on β_i, implies $-1 < \min\{\alpha - 1, \beta_i\} \leq \alpha - 1 < 0$.

Combining (1) and (2), we conclude that for $\omega \in \left(\frac{\pi}{2}, 2\pi\right) \setminus \pi$,

$$\min\{\alpha - 1, \beta_i\} \in (-1, 0) \tag{3.27}$$

For $\forall\phi \in H_0^1(\Omega)$, taking $s = -\min\{\alpha - 1, \beta_i\} \in (0, 1)$ in Lemma 3.5i yields $r^{\min\{\alpha-1,\beta_i\}}\phi = r^{-s}\phi \in L^2(\Omega)$ and

$$\|r^{\min\{\alpha-1,\beta_i\}}\phi\| = \|r^{-s}\phi\| \leq C\|\phi\|_{H^s(\Omega)} \leq C\|\phi\|_{H^1(\Omega)} \tag{3.28}$$

By taking $s' = -2\min\{\alpha - 1, \beta_i\}$ and $\gamma = -\min\{\alpha - 1, \beta_i\}$ in Lemma 3.5ii, it follows $g = r^{-s'}\phi \in H^{-\gamma}(\Omega) = H^{\min\{\alpha-1,\beta_i\}}(\Omega)$ and

$$\|g\|_{H^{\min\{\alpha-1,\beta_i\}}(\Omega)} = \|r^{-s'}\phi\|_{H^{-\gamma}(\Omega)} \leq C\|r^{-s'+\gamma}\phi\| = C\|r^{\min\{\alpha-1,\beta_i\}}\phi\| \tag{3.29}$$

(3.28) and (3.29) imply that for $\forall\phi \in H_0^1(\Omega)$,

TABLE 2 | The regularity of σ_i in different cases. (— means no such term.)

ω	$(0, \frac{\pi}{2}]$	$(\frac{\pi}{2}, \pi)$	$(\pi, \frac{3\pi}{2}]$	$(\frac{3\pi}{2}, 2\pi)$
σ_1	—	$H^{2+\beta_1}(\Omega)$	$H^{1+\alpha}(\Omega)$	$H^{1+\alpha}(\Omega)$
σ_2	—	—	$H^{2+\beta_2}(\Omega)$	$H^{1+\alpha}(\Omega)$
σ_3	—	—	—	$H^{2+\beta_3}(\Omega)$

$$\|g\|_{H^{\min\{\alpha-1,\beta_i\}}(\Omega)} \leq C\|\phi\|_{H^1(\Omega)} \tag{3.30}$$

By Lemma 3.6, the Poisson problem (2.6) admits a unique solution $z \in H^{\min\{2+\beta_i,1+\alpha\}}(\Omega)$ and

$$\|z\|_{H^{\min\{2+\beta_i,1+\alpha\}}(\Omega)} \leq C\|g\|_{H^{\min\{\alpha-1,\beta_i\}}(\Omega)} \tag{3.31}$$

which, combined with (3.29), yields the estimate (3.26). □

By (2.15) in Definition 2.4, we have $\xi_i \in H^{\beta_i} \subset H^{-1}(\Omega)$, where $-1 < \beta_i < 1 - \frac{i\pi}{\omega}$, $i = 1, \ldots, N$ satisfying $\beta_1 > \ldots > \beta_N$. Applying Lemma 3.6 to the Poisson problem (2.18), it follows $\sigma_i \in H^{\min\{2+\beta_i,1+\alpha\}}$, which is further specified in Table 2.

Then for the finite element solution $\sigma_{i,n}$ in (3.4), we have the following result.

Lemma 3.8. *For $\sigma_{i,n}$ in Algorithm 3.1, we have for $1 \leq i \leq N$,*

$$\|\sigma_i - \sigma_{i,n}\|_{H^1(\Omega)} \leq Ch^{\min\{1+\beta_i,\alpha\}} \tag{3.32a}$$

$$\|\sigma_i - \sigma_{i,n}\| \leq Ch^{\min\{1+\beta_i+\min\{\alpha,1\},2\alpha\}} \tag{3.32b}$$

$$\|r^{\min\{\alpha-1,\beta_i\}}(\sigma_j - \sigma_{j,n})\| \leq Ch^{\min\{1+\beta_i,\alpha\}+\min\{1+\beta_j,\alpha\}} \tag{3.32c}$$

where $1 \leq j \leq N$.

Proof. The difference of weak formulation of (2.18) and (3.4) gives

$$A(\sigma_i - \sigma_{i,n}, \phi) = (\xi_i - \xi_{i,n}, \phi) \tag{3.33}$$

Let $\sigma_{i,I} \in S_n$ be the nodal interpolation of σ_i. Set $\epsilon_i = \sigma_{i,I} - \sigma_i$, $e_i = \sigma_{i,I} - \sigma_{i,n}$ and take $\phi = e_i$ in the equation above, we have

$$A(e_i, e_i) = A(\epsilon_i, e_i) + (\xi_i - \xi_{i,n}, e_i),$$

which implies

$$\|e_i\|_{H^1(\Omega)} \leq \|\epsilon_i\|_{H^1(\Omega)} + \|\xi_i - \xi_{i,n}\|_{H^{-1}(\Omega)}.$$

Using the triangle inequality, it follows

$$\begin{aligned}\|\sigma_i - \sigma_{i,n}\|_{H^1(\Omega)} &\leq \|e_i\|_{H^1(\Omega)} + \|\epsilon_i\|_{H^1(\Omega)}\\ &\leq C\left(\|\epsilon_i\|_{H^1(\Omega)} + \|\xi_i - \xi_{i,n}\|_{H^{-1}(\Omega)}\right)\\ &\leq Ch^{\min\{1+\beta_i,\alpha\}},\end{aligned}$$

where we have used the projection error (3.7) and (3.9d).

To obtain the error in L^2 norm, we consider the Poisson problem (2.6). By the Aubin–Nitsche Lemma in ([10], Theorem 3.2.4), we have

$$\|\sigma_i - \sigma_{i,n}\| \le C\|\sigma_i - \sigma_{i,n}\|_{H^1(\Omega)} \sup_{g\in L^2(\Omega)} \left(\frac{\inf_{\psi\in S_n} \|z-\psi\|_{H^1(\Omega)}}{\|g\|} \right). \tag{3.34}$$

By the regularity (3.25), we have

$$\begin{aligned} \inf_{\psi\in S_n} \|z-\psi\|_{H^1(\Omega)} &\le \|z - z_I\|_{H^1(\Omega)} \\ &\le Ch^{\min\{\alpha,1\}}\|z\|_{H^{1+\min\{\alpha,1\}}(\Omega)} \\ &\le Ch^{\min\{\alpha,1\}}\|g\|. \end{aligned} \tag{3.35}$$

Plugging (3.35) and (3.32a) into (3.34) gives the estimate (3.32b).

We take $g = r^{2\min\{\alpha-1,\beta_i\}}(\sigma_j - \sigma_{j,n})$, $i = 1, \dots, N$ in (2.6), since $\sigma_j - \sigma_{j,n} \in H_0^1(\Omega)$, so Lemma 3.7indicates that $z \in H^{\min\{2+\beta_i,1+\alpha\}}(\Omega)$. By (3.7), we have the interpolation error

$$\|z - z_I\|_{H^1(\Omega)} \le Ch^{\min\{1+\beta_i,\alpha\}}\|z\|_{H^{\min\{2+\beta_i,1+\alpha\}}(\Omega)} \tag{3.36}$$

The weak formulation of (2.6) with given g is find to $z \in H_0^1(\Omega)$ such that

$$\langle r^{2\min\{\alpha-1,\beta_i\}}(\sigma_j - \sigma_{j,n}), \psi\rangle = A(z,\psi), \quad \forall \psi \in H_0^1(\Omega).$$

Set $\psi = \sigma_j - \sigma_{j,n}$ and subtract (3.33) with $\phi = z_I$ from the equation above, it follows

$$\begin{aligned} &\|r^{\min\{\alpha-1,\beta_i\}}(\sigma_j - \sigma_{j,n})\|^2 \\ &= A(\sigma_j - \sigma_{j,n}, z - z_I) + (\xi_j - \xi_{j,n}, z_I) \\ &= A(\sigma_j - \sigma_{j,n}, z - z_I) + (\xi_j - \xi_{j,n}, z_I - z) + (\xi_j - \xi_{j,n}, z) \\ &\le \|\sigma_j - \sigma_{j,n}\|_{H^1(\Omega)}\|z - z_I\|_{H^1(\Omega)} \\ &\quad + \|\xi_j - \xi_{j,n}\|_{H^{-1}(\Omega)}\big(\|z_I - z\|_{H^1(\Omega)} + \|z\|_{H^1(\Omega)}\big). \end{aligned}$$

By the estimates in (3.9d), (3.32a), (3.36), and the regularity result in Lemma 3.7, it holds

$$\begin{aligned} &\|r^{\min\{\alpha-1,\beta_i\}}(\sigma_j - \sigma_{j,n})\|^2 \\ &\le Ch^{\min\{1+\beta_i,\alpha\}+\min\{1+\beta_j,\alpha\}}\|z\|_{H^{\min\{2+\beta_i,1+\alpha\}}(\Omega)} \\ &\le Ch^{\min\{1+\beta_i,\alpha\}+\min\{1+\beta_j,\alpha\}}\|r^{\min\{\alpha-1,\beta_i\}}(\sigma_j - \sigma_{j,n})\|, \end{aligned}$$

which gives the error estimate (3.32c). □

Lemma 3.4 and Lemma 3.8 imply that $\|\xi_{i,n}\|_{H^{-1}(\Omega)}$, $\|\sigma_{i,n}\|$ and $\|\nabla\sigma_{i,n}\|$, $i = 1, \dots, N$ are uniformly bounded when $h \le h_0$ for some threshold h_0.

Lemma 3.9. *For the basis $\tilde\sigma_i$ and the corresponding finite element solution $\tilde\sigma_{i,n}$, we have*

$$\|\tilde\sigma_i - \tilde\sigma_{i,n}\|_{H^1(\Omega)} \le Ch^{\min\{1+\beta_i,\alpha\}}, \quad i = 1, \dots, N \tag{3.37}$$

Proof. By Lemma 3.8, it is obvious that

$$\|\tilde\sigma_1 - \tilde\sigma_{1,n}\|_{H^1(\Omega)} = \|\sigma_1 - \sigma_{1,n}\|_{H^1(\Omega)} \le Ch^{\min\{1+\beta_1,\alpha\}}.$$

We assume that the conclusion holds for $i \le j-1$,

$$\|\tilde\sigma_i - \tilde\sigma_{i,n}\|_{H^1(\Omega)} \le Ch^{\min\{1+\beta_i,\alpha\}} \tag{3.38}$$

A quick calculation gives that

$$\begin{aligned} \nabla\tilde\sigma_j - \nabla\tilde\sigma_{j,n} &= \nabla\sigma_j - \nabla\sigma_{j,n} \\ &\quad - \sum_{i=1}^{j-1}\left(\frac{(\nabla\sigma_j, \nabla\tilde\sigma_i)}{\|\nabla\tilde\sigma_i\|^2}\nabla\tilde\sigma_i - \frac{(\nabla\sigma_{j,n}, \nabla\tilde\sigma_{i,n})}{\|\nabla\tilde\sigma_{i,n}\|^2}\nabla\tilde\sigma_{i,n} \right) \\ &= (\nabla\sigma_j - \nabla\sigma_{j,n}) \\ &\quad - \sum_{i=1}^{j-1} \frac{\begin{array}{c}((\nabla\sigma_j, \nabla\tilde\sigma_i)\|\nabla\tilde\sigma_{i,n}\|^2\nabla\tilde\sigma_i \\ -(\nabla\sigma_{j,n}, \nabla\tilde\sigma_{i,n})\|\nabla\tilde\sigma_i\|^2\nabla\tilde\sigma_{i,n})\end{array}}{\|\nabla\tilde\sigma_i\|^2\|\nabla\tilde\sigma_{i,n}\|^2}. \end{aligned}$$

We then have

$$\begin{aligned} \|\nabla\tilde\sigma_j - \nabla\tilde\sigma_{j,n}\| &\le \|\nabla\sigma_j - \nabla\sigma_{j,n}\| \\ &\quad + \sum_{i=1}^{j-1} \frac{\left\|(\nabla\sigma_j, \nabla\tilde\sigma_i)\|\nabla\tilde\sigma_{i,n}\|^2\nabla\tilde\sigma_i - (\nabla\sigma_{j,n}, \nabla\tilde\sigma_{i,n})\|\nabla\tilde\sigma_i\|^2\nabla\tilde\sigma_{i,n}\right\|}{\|\nabla\tilde\sigma_i\|^2\|\nabla\tilde\sigma_{i,n}\|^2}. \end{aligned}$$

We know that $\nabla\tilde\sigma_i$ obtained through (3.10) depend only on Ω. Therefore, we have

$$0 < \gamma_1 \le \|\nabla\tilde\sigma_i\| \le \gamma_2, \quad i = 1, \dots, N \tag{3.39}$$

where $\gamma_1 = \min_{1\le i\le N}\{\|\nabla\tilde\sigma_i\|\}$, and $\gamma_2 = \max_{1\le i\le N}\{\|\nabla\tilde\sigma_i\|\}$. Let $h \le h_0 \le \min\left\{1, \left(\frac{\gamma_1}{2C}\right)^{\frac{1}{\min\{1+\beta_i,\alpha\}}}\right\}$, $i = 1, \dots, j-1$ in (3.37), it follows that

$$\frac{1}{2}\gamma_1 \le \|\nabla\tilde\sigma_{i,n}\| \le \gamma_2 - \frac{1}{2}\gamma_1, \quad i = 1, \dots, j-1 \tag{3.40}$$

(3.39) and (3.40) implies

$$\frac{1}{\|\nabla\tilde\sigma_i\|^2\|\nabla\tilde\sigma_{i,n}\|^2} \le C \tag{3.41}$$

where C is a constant. By Lemma 3.8, it holds

$$\|\nabla\sigma_j - \nabla\sigma_{j,n}\| \le \|\sigma_j - \sigma_{j,n}\|_{H^1(\Omega)} \le Ch^{\min\{1+\beta_j,\alpha\}} \tag{3.42}$$

To this end, we will get an error estimate for $\left\|(\nabla\sigma_j, \nabla\tilde\sigma_i)\|\nabla\tilde\sigma_{i,n}\|^2\nabla\tilde\sigma_i - (\nabla\sigma_{j,n}, \nabla\tilde\sigma_{i,n})\|\nabla\tilde\sigma_i\|^2\nabla\tilde\sigma_{i,n}\right\|$. Note that

$$\begin{aligned} &(\nabla\sigma_j, \nabla\tilde\sigma_i)\|\nabla\tilde\sigma_{i,n}\|^2\nabla\tilde\sigma_i - (\nabla\sigma_{j,n}, \nabla\tilde\sigma_{i,n})\|\nabla\tilde\sigma_i\|^2\nabla\tilde\sigma_{i,n} \\ &= (\nabla\sigma_j - \nabla\sigma_{j,n}, \nabla\tilde\sigma_i)\|\nabla\tilde\sigma_{i,n}\|^2\nabla\tilde\sigma_i \\ &\quad + (\nabla\sigma_{j,n}, \nabla\tilde\sigma_i - \nabla\tilde\sigma_{i,n})\|\nabla\tilde\sigma_{i,n}\|^2\nabla\tilde\sigma_i \\ &\quad + (\nabla\sigma_{j,n}, \nabla\tilde\sigma_{i,n})(\|\nabla\tilde\sigma_{i,n}\|^2 - \|\nabla\tilde\sigma_i\|^2)\nabla\tilde\sigma_i \\ &\quad + (\nabla\sigma_{j,n}, \nabla\tilde\sigma_{i,n})\|\nabla\tilde\sigma_i\|^2(\nabla\tilde\sigma_i - \nabla\tilde\sigma_{i,n}) \\ &:= T_1 + T_2 + T_3 + T_4. \end{aligned}$$

By Lemma 3.8 again, we have

$$\|T_1\| \le \|\nabla\tilde\sigma_{i,n}\|^2\|\nabla\tilde\sigma_i\|^2\|\nabla\sigma_j - \nabla\sigma_{j,n}\| \le Ch^{\min\{1+\beta_j,\alpha\}}.$$

By assumption (3.38), we have

$$\|T_2\| \le \|\nabla\sigma_{j,n}\|\|\nabla\tilde\sigma_{i,n}\|^2\|\nabla\tilde\sigma_i\|\|\nabla\tilde\sigma_i - \nabla\tilde\sigma_{i,n}\| \le Ch^{\min\{1+\beta_i,\alpha\}}.$$

$$\begin{aligned} \|T_3\| &\le \|\nabla\sigma_{j,n}\|\|\nabla\tilde\sigma_{i,n}\|\|\nabla\tilde\sigma_i\| \times \big(\|\nabla\tilde\sigma_i\| + \|\nabla\tilde\sigma_{i,n}\|\big) \\ &\quad \times \big|\|\nabla\tilde\sigma_i\| - \|\nabla\tilde\sigma_{i,n}\|\big| \le Ch^{\min\{1+\beta_i,\alpha\}}, \end{aligned}$$

where we used the inequality $\big|\|\nabla\tilde\sigma_i\| - \|\nabla\tilde\sigma_{i,n}\|\big| \le \|\nabla\tilde\sigma_i - \nabla\tilde\sigma_{i,n}\|$.

Last, we have

$$\|T_4\| \le \|\nabla\sigma_{j,n}\|\|\nabla\tilde\sigma_{i,n}\|\|\nabla\tilde\sigma_i\|^2\|\nabla\tilde\sigma_i - \nabla\tilde\sigma_{i,n}\| \le Ch^{\min\{1+\beta_i,\alpha\}}.$$

Thus, we have

$$\begin{aligned}&\left\|(\nabla\sigma_j,\nabla\tilde\sigma_i)\|\nabla\tilde\sigma_{i,n}\|^2\nabla\tilde\sigma_i - (\nabla\sigma_{j,n},\nabla\tilde\sigma_{i,n})\|\nabla\tilde\sigma_i\|^2\nabla\tilde\sigma_{i,n}\right\|\\ &\quad\le \sum_{l=1}^4\|T_l\| \le Ch^{\min\{1+\beta_i,\alpha\}}.\end{aligned} \tag{3.43}$$

Note that $\beta_i > \beta_j$, thus the combination of (3.42) and (3.43) gives

$$\|\tilde\sigma_j - \tilde\sigma_{j,n}\|_{H^1(\Omega)} \le Ch^{\min\{1+\beta_j,\alpha\}} \tag{3.44}$$

The method of induction leads to the conclusion. □

Lemma 3.10. *For $i = 1,\dots,N$, it holds $r^{-\min\{\alpha-1,\beta_i\}}\tilde\xi_i \in L^2(\Omega)$ and*

$$\|r^{-\min\{\alpha-1,\beta_i\}}\tilde\xi_i\| \le C \tag{3.45}$$

where C depends on β_i and Ω.

Proof. By (2.15), for $k \le i$,

$$\begin{aligned}r^{-\min\{\alpha-1,\beta_i\}}\xi_k &= \eta(r;\tau,R)r^{-\min\{\alpha-1,\beta_i\}-\frac{k\pi}{\omega}}\sin\left(\frac{k\pi}{\omega}\theta\right)\\ &\quad + r^{-\min\{\alpha-1,\beta_i\}}\zeta_k := T_{11} + T_{12}.\end{aligned}$$

Since $\beta_i \le \beta_k < 1 - \frac{k\pi}{\omega}$, it follows

$$-\min\{\alpha-1,\beta_i\} - \frac{k\pi}{\omega} > \beta_i - \min\{\alpha-1,\beta_i\} - 1 \ge -1.$$

Therefore, $T_{11} \in L^2(\Omega)$, namely,

$$\|T_{11}\| \le C \tag{3.46}$$

For T_{12}, it follows

$$\|T_{12}\| \le \|r^{-\min\{\alpha-1,\beta_i\}}\|_{L^\infty(\Omega)}\|\zeta_k\| \le C\|\zeta_k\|_{H^1(\Omega)} \tag{3.47}$$

(3.46) and (3.47) imply that

$$\|r^{-\min\{\alpha-1,\beta_i\}}\xi_k\| \le \|T_{11}\| + \|T_{12}\| \le C \tag{3.48}$$

By the construction of $\tilde\xi_i$ in (3.11), we can obtain the estimate (3.45). □

Lemma 3.11. *For $i = 1,\dots,N$, the orthogonal functions $\tilde\sigma_i$ in (3.10), $\tilde\xi_i$ in (3.11), and their finite element approximations $\tilde\sigma_{i,n}$ in (3.18), $\tilde\xi_{i,n}$ in (3.19) satisfy*

$$\|\tilde\xi_i - \tilde\xi_{i,n}\|_{H^{-1}(\Omega)} \le Ch^{\min\{1+\beta_i+\min\{\alpha,1\},2\alpha\}} \tag{3.49a}$$

$$\|\tilde\sigma_i - \tilde\sigma_{i,n}\| \le Ch^{\min\{1+\beta_i+\min\{\alpha,1\},2\alpha\}} \tag{3.49b}$$

$$\|r^{\min\{\alpha-1,\beta_k\}}(\tilde\sigma_i - \tilde\sigma_{i,n})\| \le Ch^{\min\{1+\beta_k,\alpha\}+\min\{1+\beta_i,\alpha\}} \tag{3.49c}$$

where $1 \le k \le N$.

Proof. It is easy to verify that the estimates in (3.49) hold when $i = 1$, and we assume that they also hold for $i \le j-1$ if $N \ge 2$. Next, we prove the estimates in (3.49) hold at j. The proof for (3.49b) is similar to that for (3.49a), we will skip its proof.

Using the similar argument as in Lemma 3.9, we have that $\|\tilde\xi_i\|_{H^{-1}(\Omega)}, \|\tilde\sigma_i\|, i = 1,\dots,N$ are uniformly bounded. When $h \le h_0$ for some h_0, it follows that $\|\tilde\xi_{i,n}\|_{H^{-1}(\Omega)}, \|\tilde\sigma_{i,n}\|, 1 \le i \le j-1$ are also uniformly bounded.

The difference of the $\tilde\xi_j - \tilde\xi_{j,n}$, $j = 2,\dots,N$, gives

$$\begin{aligned}\tilde\xi_j - \tilde\xi_{j,n} &= \xi_j - \xi_{j,n} - \sum_{i=1}^{j-1}\left(\frac{(\nabla\sigma_j,\nabla\tilde\sigma_i)}{\|\nabla\tilde\sigma_i\|^2}\tilde\xi_i - \frac{(\nabla\sigma_{j,n},\nabla\tilde\sigma_{i,n})}{\|\nabla\tilde\sigma_{i,n}\|^2}\tilde\xi_{i,n}\right)\\ &= (\xi_j - \xi_{j,n}) - \sum_{i=1}^{j-1}\frac{\begin{pmatrix}(\nabla\sigma_j,\nabla\tilde\sigma_i)\|\nabla\tilde\sigma_{i,n}\|^2\tilde\xi_i\\ -(\nabla\sigma_{j,n},\nabla\tilde\sigma_{i,n})\|\nabla\tilde\sigma_i\|^2\tilde\xi_{i,n}\end{pmatrix}}{\|\nabla\tilde\sigma_i\|^2\|\nabla\tilde\sigma_{i,n}\|^2}.\end{aligned} \tag{3.50}$$

By (3.41), $\frac{1}{\|\nabla\tilde\sigma_i\|^2\|\nabla\tilde\sigma_{i,n}\|^2}$ are uniformly bounded. We denote by

$$\begin{aligned}&(\nabla\sigma_j,\nabla\tilde\sigma_i)\|\nabla\tilde\sigma_{i,n}\|^2\tilde\xi_i - (\nabla\sigma_{j,n},\nabla\tilde\sigma_{i,n})\|\nabla\tilde\sigma_i\|^2\tilde\xi_{i,n}\\ &\quad= (\nabla\sigma_j - \nabla\sigma_{j,n},\nabla\tilde\sigma_i)\|\nabla\tilde\sigma_{i,n}\|^2\tilde\xi_i\\ &\qquad+ (\nabla\sigma_{j,n},\nabla\tilde\sigma_i - \nabla\tilde\sigma_{i,n})\|\nabla\tilde\sigma_{i,n}\|^2\tilde\xi_i\\ &\qquad+ (\nabla\sigma_{j,n},\nabla\tilde\sigma_{i,n})(\|\nabla\tilde\sigma_{i,n}\|^2 - \|\nabla\tilde\sigma_i\|^2)\tilde\xi_i\\ &\qquad+ (\nabla\sigma_{j,n},\nabla\tilde\sigma_{i,n})\|\nabla\tilde\sigma_i\|^2(\tilde\xi_i - \tilde\xi_{i,n})\\ &\quad:= T_1 + T_2 + T_3 + T_4.\end{aligned}$$

By (3.50), it follows

$$\begin{aligned}\|\tilde\xi_j - \tilde\xi_{j,n}\|_{H^{-1}(\Omega)} &\le \|\xi_j - \xi_{j,n}\|_{H^{-1}(\Omega)}\\ &\quad+ \sum_{i=1}^{j-1}\frac{\left\|(\nabla\sigma_j,\nabla\tilde\sigma_i)\|\nabla\tilde\sigma_{i,n}\|^2\tilde\xi_i - (\nabla\sigma_{j,n},\nabla\tilde\sigma_{i,n})\|\nabla\tilde\sigma_i\|^2\tilde\xi_{i,n}\right\|_{H^{-1}(\Omega)}}{\|\nabla\tilde\sigma_i\|^2\|\nabla\tilde\sigma_{i,n}\|^2}.\end{aligned}$$

From (3.9d), we have

$$\|\xi_j - \xi_{j,n}\|_{H^{-1}(\Omega)} \le Ch^{2\min\{\alpha,1\}} \tag{3.51}$$

By taking $\phi = \sigma_j - \sigma_{j,n} \in H_0^1(\Omega)$ in (3.13), we have

$$(\nabla\sigma_j - \nabla\sigma_{j,n},\nabla\tilde\sigma_i) = \langle\sigma_j - \sigma_{j,n},\tilde\xi_i\rangle \tag{3.52}$$

which implies that

$$\begin{aligned}T_1 &= \langle\sigma_j - \sigma_{j,n},\tilde\xi_i\rangle\|\nabla\tilde\sigma_{i,n}\|^2\tilde\xi_i\\ &= \langle r^{\min\{\alpha-1,\beta_i\}}(\sigma_j - \sigma_{j,n}), r^{-\min\{\alpha-1,\beta_i\}}\tilde\xi_i\rangle\|\nabla\tilde\sigma_{i,n}\|^2\tilde\xi_i.\end{aligned}$$

By Lemma 3.10, we have $r^{-\min\{\alpha-1,\beta_i\}}\tilde\xi_i \in L^2(\Omega)$. Therefore, we have the estimate

$$\begin{aligned}\|T_1\|_{H^{-1}(\Omega)} &\le \|r^{\min\{\alpha-1,\beta_i\}}(\sigma_j - \sigma_{j,n})\|\|r^{-\min\{\alpha-1,\beta_i\}}\tilde\xi_i\|\|\nabla\tilde\sigma_{i,n}\|^2\|\tilde\xi_i\|_{H^{-1}(\Omega)}\\ &\le Ch^{\min\{1+\beta_i,\alpha\}+\min\{1+\beta_j,\alpha\}}\\ &= Ch^{\min\{1+\beta_j+\min\{\alpha,1\},2\alpha\}},\end{aligned}$$

where we have used the estimate (3.32c).

Subtracting Equation (3.20) from Equation (3.13) and setting $\phi = \sigma_{j,n}$ yields

$$(\nabla\sigma_{j,n}, \nabla\tilde{\sigma}_i - \nabla\tilde{\sigma}_{i,n}) = \langle \sigma_{j,n}, \tilde{\xi}_i - \tilde{\xi}_{i,n}\rangle.$$

Thus, we have by the assumption,

$$\begin{aligned}\|T_2\|_{H^{-1}(\Omega)} &\le \|\sigma_{j,n}\|\|\tilde{\xi}_i - \tilde{\xi}_{i,n}\|_{H^{-1}(\Omega)}\|\nabla\tilde{\sigma}_{i,n}\|^2\|\tilde{\xi}_i\|_{H^{-1}(\Omega)} \\ &\le Ch^{\min\{1+\beta_i+\min\{\alpha,1\},2\alpha\}}.\end{aligned}$$

We have by (3.13) and (3.20),

$$\begin{aligned}\|\nabla\tilde{\sigma}_{i,n}\|^2 - \|\nabla\tilde{\sigma}_i\|^2 &= \langle \tilde{\sigma}_{i,n}, \tilde{\xi}_{i,n}\rangle - \langle \tilde{\sigma}_i, \tilde{\xi}_i\rangle \\ &= \langle \tilde{\sigma}_{i,n} - \tilde{\sigma}_i, \tilde{\xi}_{i,n}\rangle + \langle \tilde{\sigma}_i, \tilde{\xi}_{i,n} - \tilde{\xi}_i\rangle := T_{31} + T_{32}.\end{aligned}$$

By the assumption for (3.49c), we have

$$\begin{aligned}|T_{31}| &= |\langle \tilde{\sigma}_{i,n} - \tilde{\sigma}_i, \tilde{\xi}_{i,n}\rangle| \\ &\le \|r^{\min\{\alpha-1,\beta_i\}}(\tilde{\sigma}_{i,n} - \tilde{\sigma}_i)\|\|r^{-\min\{\alpha-1,\beta_i\}}\tilde{\xi}_{i,n}\| \\ &\le Ch^{\min\{1+\beta_i+\min\{\alpha,1\},2\alpha\}}.\end{aligned}$$

For the second term, we have by the assumption for (3.49a),

$$\begin{aligned}|T_{32}| = |\langle \tilde{\sigma}_i, \tilde{\xi}_{i,n} - \tilde{\xi}_i\rangle| &\le \|\tilde{\sigma}_i\|_{H^1(\Omega)}\|\tilde{\xi}_{i,n} - \tilde{\xi}_i\|_{H^{-1}(\Omega)} \\ &\le Ch^{\min\{1+\beta_i+\min\{\alpha,1\},2\alpha\}}.\end{aligned}$$

The estimates of $|T_{31}|$ and $|T_{32}|$ imply that

$$\|T_3\|_{H^{-1}(\Omega)} \le Ch^{\min\{1+\beta_i+\min\{\alpha,1\},2\alpha\}}.$$

Again by the assumption for (3.49a), we have

$$\begin{aligned}\|T_4\|_{H^{-1}(\Omega)} &\le \|\nabla\sigma_{j,n}\|\|\nabla\tilde{\sigma}_{i,n}\|\|\nabla\tilde{\sigma}_i\|^2\|\tilde{\xi}_i - \tilde{\xi}_{i,n}\|_{H^{-1}(\Omega)} \\ &\le Ch^{\min\{1+\beta_i+\min\{\alpha,1\},2\alpha\}}.\end{aligned}$$

Note that $\beta_i > \beta_j$, we have

$$\sum_{l=1}^{4}\|T_l\|_{H^{-1}(\Omega)} \le Ch^{\min\{1+\beta_j+\min\{\alpha,1\},2\alpha\}} \tag{3.53}$$

The combination of (3.51) and (3.53) indicate that (3.49a) holds at j, so that the method of induction state that (3.49a) holds for $i = 1, \dots, N$.

Next, we prove the estimate (3.49c) holds at j. For $j = 2, \dots, N$, we have

$$\begin{aligned}r^{\min\{\alpha-1,\beta_k\}}(\tilde{\sigma}_j - \tilde{\sigma}_{j,n}) &= r^{\min\{\alpha-1,\beta_k\}}(\sigma_j - \sigma_{j,n}) \\ &\quad - r^{\min\{\alpha-1,\beta_k\}}\sum_{i=1}^{j-1}\left(\frac{(\nabla\sigma_j, \nabla\tilde{\sigma}_i)}{\|\nabla\tilde{\sigma}_i\|^2}\tilde{\sigma}_i - \frac{(\nabla\sigma_{j,n}, \nabla\tilde{\sigma}_{i,n})}{\|\nabla\tilde{\sigma}_{i,n}\|^2}\tilde{\sigma}_{i,n}\right) \\ &= r^{\min\{\alpha-1,\beta_k\}}\left(\sigma_j - \sigma_{j,n}\right) \\ &\quad - \sum_{i=1}^{j-1}\frac{\begin{array}{c}\left((\nabla\sigma_j, \nabla\tilde{\sigma}_i)\|\nabla\tilde{\sigma}_{i,n}\|^2 r^{\min\{\alpha-1,\beta_k\}}\tilde{\sigma}_i\right. \\ \left.-(\nabla\sigma_{j,n}, \nabla\tilde{\sigma}_{i,n})\|\nabla\tilde{\sigma}_i\|^2 r^{\min\{\alpha-1,\beta_k\}}\tilde{\sigma}_{i,n}\right)\end{array}}{\|\nabla\tilde{\sigma}_i\|^2\|\nabla\tilde{\sigma}_{i,n}\|^2},\end{aligned} \tag{3.54}$$

where $1 \le k \le N$. By (3.41), $\frac{1}{\|\nabla\tilde{\sigma}_i\|^2\|\nabla\tilde{\sigma}_{i,n}\|^2}$ are uniformly bounded. We denote by

$$\begin{aligned}&(\nabla\sigma_j, \nabla\tilde{\sigma}_i)\|\nabla\tilde{\sigma}_{i,n}\|^2 r^{\min\{\alpha-1,\beta_k\}}\tilde{\sigma}_i - (\nabla\sigma_{j,n}, \nabla\tilde{\sigma}_{i,n})\|\nabla\tilde{\sigma}_i\|^2 r^{\min\{\alpha-1,\beta_k\}}\tilde{\sigma}_{i,n} \\ &= (\nabla\sigma_j - \nabla\sigma_{j,n}, \nabla\tilde{\sigma}_i)\|\nabla\tilde{\sigma}_{i,n}\|^2 r^{\min\{\alpha-1,\beta_k\}}\tilde{\sigma}_i \\ &\quad + (\nabla\sigma_{j,n}, \nabla\tilde{\sigma}_i - \nabla\tilde{\sigma}_{i,n})\|\nabla\tilde{\sigma}_{i,n}\|^2 r^{\min\{\alpha-1,\beta_k\}}\tilde{\sigma}_i \\ &\quad + (\nabla\sigma_{j,n}, \nabla\tilde{\sigma}_{i,n})(\|\nabla\tilde{\sigma}_{i,n}\|^2 - \|\nabla\tilde{\sigma}_i\|^2) r^{\min\{\alpha-1,\beta_k\}}\tilde{\sigma}_i \\ &\quad + (\nabla\sigma_{j,n}, \nabla\tilde{\sigma}_{i,n})\|\nabla\tilde{\sigma}_i\|^2 r^{\min\{\alpha-1,\beta_k\}}(\tilde{\sigma}_i - \tilde{\sigma}_{i,n}) \\ &:= K_1 + K_2 + K_3 + K_4.\end{aligned}$$

By (3.54), it follows

$$\|r^{\min\{\alpha-1,\beta_k\}}(\tilde{\sigma}_j - \tilde{\sigma}_{j,n})\| \le \|r^{\min\{\alpha-1,\beta_k\}}\left(\sigma_j - \sigma_{j,n}\right)\| + \sum_{i=1}^{j-1}\frac{\left\|\begin{array}{c}(\nabla\sigma_j, \nabla\tilde{\sigma}_i)\|\nabla\tilde{\sigma}_{i,n}\|^2 r^{\min\{\alpha-1,\beta_k\}}\tilde{\sigma}_i \\ -(\nabla\sigma_{j,n}, \nabla\tilde{\sigma}_{i,n})\|\nabla\tilde{\sigma}_i\|^2 r^{\min\{\alpha-1,\beta_k\}}\tilde{\sigma}_{i,n}\end{array}\right\|}{\|\nabla\tilde{\sigma}_i\|^2\|\nabla\tilde{\sigma}_{i,n}\|^2},$$

From (3.32b), we have

$$\|r^{\min\{\alpha-1,\beta_k\}}\left(\sigma_j - \sigma_{j,n}\right)\| \le Ch^{\min\{1+\beta_k,\alpha\}+\min\{1+\beta_j,\alpha\}} \tag{3.55}$$

Similar to the estimate of T_1, we have by (3.52),

$$K_1 = \langle r^{\min\{\alpha-1,\beta_i\}}(\sigma_j - \sigma_{j,n}), r^{-\min\{\alpha-1,\beta_i\}}\tilde{\xi}_i\rangle\|\nabla\tilde{\sigma}_{i,n}\|^2 r^{\min\{\alpha-1,\beta_k\}}\tilde{\sigma}_i.$$

Since $\tilde{\sigma}_i \in H_0^1(\Omega) \cap H^{\min\{2+\beta_i,1+\alpha\}}(\Omega)$, so we have $r^{\min\{\alpha-1,\beta_k\}}\tilde{\sigma}_i \in L^2(\Omega)$ by Lemma 3.5 or from (3.28). Therefore, we have the estimate

$$\begin{aligned}\|K_1\| &\le \|r^{\min\{\alpha-1,\beta_i\}}(\sigma_j - \sigma_{j,n})\|\|r^{-\min\{\alpha-1,\beta_i\}}\tilde{\xi}_i\|\|\nabla\tilde{\sigma}_{i,n}\|^2 \\ &\quad \|r^{\min\{\alpha-1,\beta_k\}}\tilde{\sigma}_i\| \le Ch^{\min\{1+\beta_i,\alpha\}+\min\{1+\beta_j,\alpha\}} \\ &= Ch^{\min\{1+\beta_j+\min\{\alpha,1\},2\alpha\}},\end{aligned}$$

the last equality is due to the fact that $\min\{1+\beta_i,\alpha\} = \alpha$ when $1 \le i < N$. Similar to the estimate of T_2, we have

$$\begin{aligned}\|K_2\|_{H^{-1}(\Omega)} &\le \|\sigma_{j,n}\|\|\tilde{\xi}_i - \tilde{\xi}_{i,n}\|_{H^{-1}(\Omega)}\|\nabla\tilde{\sigma}_{i,n}\|^2 \\ &\quad \cdot \|r^{\min\{\alpha-1,\beta_k\}}\tilde{\sigma}_i\| \le Ch^{\min\{1+\beta_i+\min\{\alpha,1\},2\alpha\}}.\end{aligned}$$

The estimates of $|T_{31}|$ and $|T_{32}|$ above also indicate that

$$\|K_3\| \le Ch^{\min\{1+\beta_i+\min\{\alpha,1\},2\alpha\}}.$$

By the assumption for (3.49c), we have

$$\begin{aligned}\|K_4\| &\le \|\nabla\sigma_{j,n}\|\|\nabla\tilde{\sigma}_{i,n}\|\|\nabla\tilde{\sigma}_i\|^2\|r^{\min\{\alpha-1,\beta_k\}}(\tilde{\sigma}_i - \tilde{\sigma}_{i,n})\| \\ &\le Ch^{\min\{1+\beta_k,\alpha\}+\min\{1+\beta_i,\alpha\}}.\end{aligned}$$

Note again that $\beta_i > \beta_j$, we have

$$\sum_{l=1}^{4}\|K_l\| \le Ch^{\min\{1+\beta_k,\alpha\}+\min\{1+\beta_j,\alpha\}} \tag{3.56}$$

The combination of (3.55) and (3.56) indicate that (3.49c) holds at j, so that the method of induction state that (3.49c) holds for $i = 1, \dots, N$. □

Note that $v \in H^{1+\alpha}(\Omega)$, then we have the following estimates for v_n in (3.2).

Lemma 3.12. *Let $v_n \in S_n$ be the finite element approximation to (3.2), and v be the solution to the Poisson equation in the mixed formulation (2.42). Then it follows*

$$\|v - v_n\|_{H^1(\Omega)} \le Ch^{\min\{\alpha,1\}} \tag{3.57a}$$

$$\|v - v_n\| \le Ch^{2\min\{\alpha,1\}} \tag{3.57b}$$

Proof. Subtracting (2.43b) from (3.2) gives the Galerkin orthogonality

$$A(v - v_n, \psi) = (w - w_n, \psi) \tag{3.58}$$

Let $v_I \in S_n$ be the nodal interpolation of v. Set $\epsilon = v_I - u$, $e = v_I - v_n$ and take $\psi = e$ in the equation above, we have

$$A(e, e) = A(\epsilon, e) + (w - w_n, e),$$

which implies

$$\|e\|_{H^1(\Omega)} \le \|\epsilon\|_{H^1(\Omega)} + \|w - w_n\|_{H^{-1}(\Omega)},$$

Using the triangle inequality, it follows

$$\begin{aligned}\|v - v_n\|_{H^1(\Omega)} &\le \|e\|_{H^1(\Omega)} + \|\epsilon\|_{H^1(\Omega)} \\ &\le C\left(\|\epsilon\|_{H^1(\Omega)} + \|w - w_n\|_{H^{-1}(\Omega)}\right), \\ &\le C\left(\|\epsilon\|_{H^1(\Omega)} + \|w - w_n\|\right) \le Ch^{\min\{\alpha,1\}},\end{aligned}$$

where we have used the projection error (3.7, 3.9b). To obtain the L^2 error, we consider the problem (2.6) with $g = v - v_n$, then we have

$$\|v - v_n\|^2 = A(v - v_n, z).$$

Subtract (3.58) from the above equation and set $\psi = z_I$, we have

$$\begin{aligned}\|v - v_n\|^2 &= A(v - v_n, z - z_I) + (w - w_n, z_I) \\ &= A(v - v_n, z - z_I) + (w - w_n, z_I - z) + (w - w_n, z) \\ &\le \|v - v_n\|_{H^1(\Omega)}\|z - z_I\|_{H^1(\Omega)} + \|w - w_n\|\|z - z_I\| \\ &\quad + \|w - w_n\|\|z\| \\ &\le Ch^{2\min\{\alpha,1\}}\|z\|_{H^{1+\min\{\alpha,1\}}(\Omega)} \le Ch^{2\min\{\alpha,1\}}\|v - v_n\|,\end{aligned} \tag{3.59}$$

where in the last inequality we have use the estimates (3.7, 3.9b, 3.57). By the regularity (3.25), we have

$$\|z\|_{H^{1+\min\{\alpha,1\}}(\Omega)} \le C\|v - v_n\|_{H^{\min\{\alpha,1\}-1}(\Omega)} \le C\|v - v_n\| \tag{3.60}$$

(3.59) and (3.60) give the L^2 error estimate (3.57b). □

Next, we carry out the error estimate for the finite element approximation u_n in (3.6).

Theorem 3.13. *Let $u_n \in S_n$ be the finite element approximation to (3.6), and u be the solution to the sixth-order problem (2.2). Then it follows*

$$\|u - u_n\|_{H^1(\Omega)} \le C_0 h + \sum_{i=1}^{N} C_i h^{\min\{2(1+\beta_i),1\}} \le Ch^{\gamma} \tag{3.61}$$

where $-1 < \beta_i < 1 - \frac{i\pi}{\omega}$, the convergence rate $\gamma = 1$ if $N = 0$, and$\gamma = \min\{2(1+\beta_N), 1\}$ if $1 \le N \le 3$, the constants C, C_i depend on the coefficients $\tilde{c}_i$ in (3.15).

Proof. Subtracting (3.22) from (3.17) gives

$$\begin{aligned}A(u - u_n, \tau) &= (v - v_n, \tau) - \sum_{i=1}^{N}\left(\tilde{c}_i\tilde{\sigma}_i - \tilde{c}_{i,n}\tilde{\sigma}_{i,n}, \tau\right) \\ &= (v - v_n, \tau) + \sum_{i=1}^{N}\Big[\tilde{c}_{i,n}(\tilde{\sigma}_{i,n} - \tilde{\sigma}_i, \tau) \\ &\quad + (\tilde{c}_{i,n} - \tilde{c}_i)(\tilde{\sigma}_i, \tau)\Big].\end{aligned} \tag{3.62}$$

Let $u_I \in S_n$ be the nodal interpolation of u. Set $\epsilon = u_I - u$, $e = u_I - u_n$ and take $\tau = e$ in (3.62), we have

$$\begin{aligned}A(e, e) &= A(\epsilon, e) + (v - v_n, e) \\ &\quad + \sum_{i=1}^{N}\left[\tilde{c}_{i,n}(\tilde{\sigma}_{i,n} - \tilde{\sigma}_i, e) + (\tilde{c}_{i,n} - \tilde{c}_i)(\tilde{\sigma}_i, e)\right].\end{aligned}$$

Thus, we have

$$\begin{aligned}\|e\|_{H^1(\Omega)} &\le C\big(\|\epsilon\|_{H^1(\Omega)} + \|v - v_n\|_{H^{-1}(\Omega)} \\ &\quad + \sum_{i=1}^{N}\big[|\tilde{c}_{i,n}|\|\tilde{\sigma}_i - \tilde{\sigma}_{i,n}\|_{H^{-1}(\Omega)} \\ &\quad + |\tilde{c}_i - \tilde{c}_{i,n}|\|\tilde{\sigma}_i\|_{H^{-1}(\Omega)}\big]\big).\end{aligned}$$

Using the triangle inequality and the inequality above, we have

$$\begin{aligned}&\|u - u_n\|_{H^1(\Omega)} \le \|e\|_{H^1(\Omega)} + \|\epsilon\|_{H^1(\Omega)} \\ &\le C\left(\|\epsilon\|_{H^1(\Omega)} + \|v - v_n\|_{H^{-1}(\Omega)} + \sum_{i=1}^{N}\left[|\tilde{c}_{i,n}|\|\tilde{\sigma}_i - \tilde{\sigma}_{i,n}\|_{H^{-1}(\Omega)}\right.\right. \\ &\quad \left.\left. + |\tilde{c}_i - \tilde{c}_{i,n}|\|\tilde{\sigma}_i\|_{H^{-1}(\Omega)}\right]\right).\end{aligned} \tag{3.63}$$

We shall estimate every term in (3.63). Recall the solution $u \in H^3(\Omega)$. By the interpolation error estimate (3.7),

$$\|\epsilon\|_{H^1(\Omega)} = \|u - u_I\|_{H^1(\Omega)} \le Ch\|u\|_{H^2(\Omega)} \tag{3.64}$$

Recall that $\frac{\pi}{\omega} > \frac{1}{2}$. Thus, choosing $\alpha = 1/2 < \frac{\pi}{\omega}$ in (3.57b), we have

$$\|v - v_n\|_{H^{-1}(\Omega)} \le \|v - v_n\| \le Ch \tag{3.65}$$

By (3.49b), we have

$$\|\tilde{\sigma}_i - \tilde{\sigma}_{i,n}\|_{H^{-1}(\Omega)} \le \|\tilde{\sigma}_i - \tilde{\sigma}_{i,n}\| \le Ch^{\min\{1+\beta_i+\min\{\alpha,1\},2\alpha\}}.$$

To obtain the error estimate for the third term in (3.63), we still need to show that $|\tilde{c}_{i,n}|$ is uniformly bounded. By (3.21), we have

$$\begin{aligned}|\tilde{c}_{i,n}| &= \left|\frac{\langle v_n, \tilde{\xi}_{i,n}\rangle}{\langle\tilde{\sigma}_{i,n}, \tilde{\xi}_{i,n}\rangle}\right| = \left|\frac{(\nabla\tilde{\sigma}_{i,n}, \nabla v_n)}{(\nabla\tilde{\sigma}_{i,n}, \nabla\tilde{\sigma}_{i,n})}\right| \\ &\le \frac{\|v_n\|_{H^1(\Omega)}\|\tilde{\sigma}_{i,n}\|_{H^1(\Omega)}}{\|\tilde{\sigma}_{i,n}\|^2_{H^1(\Omega)}} \le \frac{\|v_n\|_{H^1(\Omega)}}{\|\tilde{\sigma}_{i,n}\|_{H^1(\Omega)}},\end{aligned} \tag{3.66}$$

where we have used Hölder's inequality. By the regularity result (2.45) and the estimate (3.57), we have $\|v_n\|_{H^1(\Omega)} \le C\|f\|$ when $h \le h_0$ for some h_0, which together with (3.40) implies that (3.66) is uniformly bounded.

Subtracting (3.21) from (3.15) or (3.16) gives

$$\tilde{c}_i - \tilde{c}_{i,n} = \frac{(\nabla v, \nabla(\tilde{\sigma}_i - \tilde{\sigma}_{i,n}))}{\|\nabla \tilde{\sigma}_i\|^2} + \frac{(\nabla \tilde{\sigma}_{i,n}, \nabla(v - v_n))}{\|\nabla \tilde{\sigma}_i\|^2} + \frac{\|\nabla \tilde{\sigma}_{i,n}\|^2 - \|\nabla \tilde{\sigma}_i\|^2}{\|\nabla \tilde{\sigma}_i\|^2 \|\nabla \tilde{\sigma}_{i,n}\|^2}(\nabla v_n, \nabla \tilde{\sigma}_{i,n}) := T_1 + T_2 + T_3.$$

By setting $\psi = (\tilde{\sigma}_i - \tilde{\sigma}_{i,n}) \in H_0^1(\Omega)$ in (2.43), we obtain

$$(\nabla v, \nabla(\tilde{\sigma}_i - \tilde{\sigma}_{i,n})) = (w, \tilde{\sigma}_i - \tilde{\sigma}_{i,n}).$$

Thus, we have by (3.49)

$$\|T_1\| \le \frac{\|w\|}{\|\nabla \tilde{\sigma}_i\|^2}\|\tilde{\sigma}_i - \tilde{\sigma}_{i,n}\| \le Ch^{\min\{1+\beta_i+\min\{\alpha,1\},2\alpha\}}.$$

Subtracting Equation (3.2) from Equation (2.43b) and setting $\psi = \tilde{\sigma}_{i,n}$, we obtain

$$(\nabla \tilde{\sigma}_{i,n}, \nabla(v - v_n)) = (w - w_n, \nabla \sigma_{i,n}).$$

Then we have by (3.9b) and taking $\alpha = \frac{1}{2}$,

$$\|T_2\| \le \frac{1}{\|\nabla \tilde{\sigma}_i\|}\|w - w_n\| \le Ch^{2\min\{\alpha,1\}} = Ch.$$

Note that

$$\begin{aligned}\|\nabla \tilde{\sigma}_{i,n}\|^2 - \|\nabla \tilde{\sigma}_i\|^2 &= (\nabla \tilde{\sigma}_{i,n} - \nabla \tilde{\sigma}_i, \nabla \tilde{\sigma}_{i,n}) + (\nabla \tilde{\sigma}_{i,n} - \nabla \tilde{\sigma}_i, \nabla \tilde{\sigma}_i) \\ &= \langle \tilde{\xi}_{i,n} - \tilde{\xi}_i, \tilde{\sigma}_{i,n}\rangle + \langle \tilde{\sigma}_{i,n} - \tilde{\sigma}_i, \tilde{\xi}_i\rangle.\end{aligned}$$

By (3.49a), we have

$$\|\tilde{\xi}_i - \tilde{\xi}_{i,n}\|_{H^{-1}(\Omega)} \le Ch^{\min\{1+\beta_i+\min\{\alpha,1\},2\alpha\}} \tag{3.67}$$

It is easy to check

$$\begin{aligned}|\langle \tilde{\xi}_{i,n} - \tilde{\xi}_i, \tilde{\sigma}_{i,n}\rangle| &\le \|\tilde{\xi}_{i,n} - \tilde{\xi}_i\|_{H^{-1}(\Omega)}\|\tilde{\sigma}_{i,n}\|_{H^1(\Omega)} \\ &\le Ch^{\min\{1+\beta_i+\min\{\alpha,1\},2\alpha\}}.\end{aligned}$$

Note that

$$\begin{aligned}|\langle \tilde{\sigma}_{i,n} - \tilde{\sigma}_i, \tilde{\xi}_i\rangle| &= |\langle r^{\min\{\alpha-1,\beta_i\}}(\tilde{\sigma}_i - \tilde{\sigma}_{i,n}), r^{-\min\{\alpha-1,\beta_i\}}\tilde{\xi}_i\rangle| \\ &\le \|r^{\min\{\alpha-1,\beta_i\}}(\tilde{\sigma}_i - \tilde{\sigma}_{i,n})\|\|r^{-\min\{\alpha-1,\beta_i\}}\tilde{\xi}_i\| \\ &\le Ch^{2\min\{1+\beta_i,\alpha\}}.\end{aligned}$$

The last two inequalities imply that

$$\|T_3\| \le Ch^{2\min\{1+\beta_i,\alpha\}}.$$

Thus, we have

$$|\tilde{c}_i - \tilde{c}_{i,n}| \le \sum_{l=1}^{3}\|T_l\| \le Ch^{2\min\{1+\beta_i,\alpha\}} \tag{3.68}$$

TABLE 3 | The value of $\min\{2(1+\beta_i), 1\}$ and γ in Section 3.13 for different ω.

ω	$(0, \frac{\pi}{2}]$	$(\frac{\pi}{2}, \frac{2\pi}{3}]$	$(\frac{2\pi}{3}, \pi)$	$(\pi, \frac{4\pi}{3}]$	$(\frac{4\pi}{3}, \frac{3\pi}{2}]$	$(\frac{3\pi}{2}, 2\pi)$
$\min\{2(1+\beta_1), 1\}$	—	$2(1+\beta_1)$	1	1	1	1
$\min\{2(1+\beta_2), 1\}$	—	—	—	$2(1+\beta_2)$	1	1
$\min\{2(1+\beta_3), 1\}$	—	—	—	—	—	$2(1+\beta_3)$
γ	1	$2(1+\beta_1)$	1	$2(1+\beta_2)$	1	$2(1+\beta_3)$

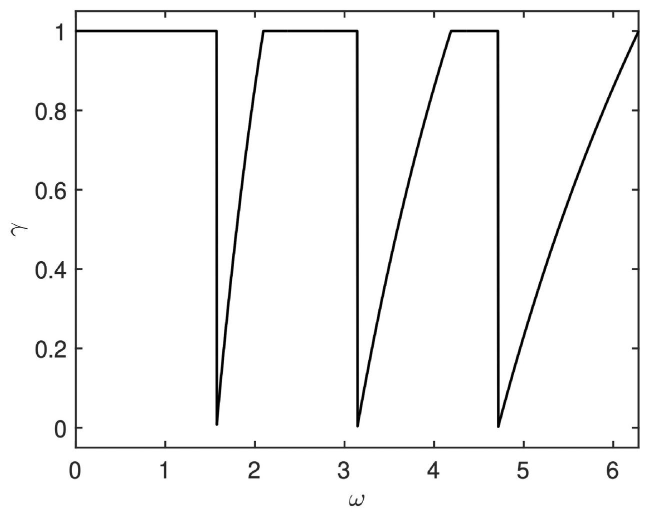


FIGURE 2 | The H^1 convergence rate γ in Section 3.13 for different ω.

Plugging (3.64), (3.65), (3.67) and (3.68) with $\alpha = \frac{1}{2}$ into (3.63), the conclusion holds. □

Remark 3.14. For the following cases, we have $\min\{2(1+\beta_i), 1\} = 1$, (i) $1 \le i < N$; (ii) $i = N$ and $\beta_i \ge -\frac{1}{2}$. To better view $\|u - u_n\|_{H^1(\Omega)}$ in (3.61), we explicitly show the value of $\min\{2(1+\beta_i), 1\}$ and the value of γ in Table 3 and Figure 2.

4 | Numerical Illustrations

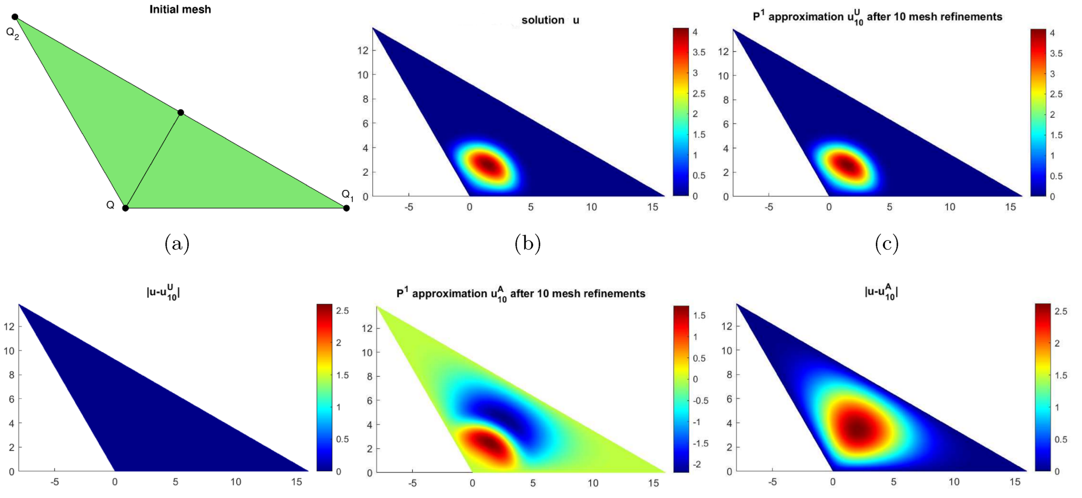
In this section, we present numerical test results to validate our theoretical predictions for Algorithm 3.1 solving the sixth-order problem (1.1). For comparison, we also implement the finite element method for the direct mixed formulation (2.4), referred to as the *direct mixed finite element method*. We will utilize the following convergence rate as an indicator of the actual convergence rate of the exact solutions u, v, w in (2.42) are given, then calculate the convergence rate by

$$\mathcal{R} = \log_2 \frac{|\phi - \phi_{j-1}|_{H^1(\Omega)}}{|\phi - \phi_j|_{H^1(\Omega)}} \tag{4.1}$$

otherwise,

$$\mathcal{R} = \log_2 \frac{|\phi_j - \phi_{j-1}|_{H^1(\Omega)}}{|\phi_{j+1} - \phi_j|_{H^1(\Omega)}} \tag{4.2}$$

Here, ϕ_j represents the finite element solution on the mesh $\mathcal{T}_j$, obtained after j refinements of the initial triangulation $\mathcal{T}_0$. It can be either u_j, v_j, or w_j, depending on the underlying Poisson



(a) (b) (c)

(d) (e) (f)

FIGURE 3 | Example 4.1 Test case 1: (a) the domain and the initial mesh; (b) the "spurious solution" u; (c) the direct mixed finite element solution u_{10}^U; (d) the difference $|u-u_{10}^U|$; (e) the solution u_{10}^A from Algorithm 3.1; (f) the difference $|u-u_{10}^A|$.

problem. In particular, suppose the actual convergence rate is $|\phi - \phi_j|_{H^1(\Omega)} = O(h^\beta)$ for $\beta > 0$. Then, for the P_1 finite element method, the rate in (4.2) is also a good approximation of the exponent β as the level of refinements j increases [30].

We use the following cut-off function in Algorithm 3.1:

$$\eta(r;\tau,R) = \begin{cases} 0, & \text{if } r \geq R, \\ 1, & \text{if } r \leq \tau R, \\ \frac{1}{2} - \frac{15}{16}\left(\frac{2r}{R(1-\tau)} - \frac{1+\tau}{1-\tau}\right) \\ \quad + \frac{5}{8}\left(\frac{2r}{R(1-\tau)} - \frac{1+\tau}{1-\tau}\right)^3 \\ \quad - \frac{3}{16}\left(\frac{2r}{R(1-\tau)} - \frac{1+\tau}{1-\tau}\right)^5, & \text{otherwise.} \end{cases}$$

We set the default parameters $R = \frac{32}{5}, \tau = \frac{1}{8}$. If a different R is used, it will be specified.

Example 4.1. We solve the problem (1.1) on different domains using both the direct mixed finite element method and Algorithm 3.1on quasi-uniform meshes obtained by midpoint refinements with the given initial mesh. We start with a "wrong solution" $u \notin H^3(\Omega)$,

$$u(r,\theta) = \tilde{\eta}(r;\tau,R) r^{\frac{\pi}{\omega}} \sin\left(\frac{\pi}{\omega}\theta\right) \tag{4.3}$$

where $\tilde{\eta}(r;\tau,R)$ is also a cut-off function

$$\tilde{\eta}(r;\tau,R) = \begin{cases} 0, & \text{if } r > R, \\ 1, & \text{if } r < \tau R, \\ \frac{1}{2} + \sum_{i=0}^{6} C_i \left(\frac{2r}{R(1-\tau)} - \frac{1+\tau}{1-\tau}\right)^{2i+1}, & \text{otherwise,} \end{cases} \tag{4.4}$$

with $R = \frac{32}{5}, \tau = \frac{1}{8}$, and the coefficients C_i are determined by solving the linear system

$$\tilde{\eta}^{(i)}(R;\tau,R) = 0, \quad i = 0, \ldots, 6.$$

The source term f is obtained by calculating

$$f = -\Delta(\Delta(\Delta u)),$$

and it can be verified that $f \in L^2(\Omega)$. Note that $u \notin H^3(\Omega)$ and therefore u is not the solution of the weak formulation (2.2) because the "true solution" should be a function in $H^3(\Omega)$. The purpose of this example is to test the convergence of the finite element method for the direct mixed formulation and Algorithm 3.1 to the "spurious solution" u in (4.3).

Test case 1. Take Ω as the triangle $\triangle QQ_1Q_2$ with $Q(0,0)$, $Q_1(16,0)$ and $Q_2(-8, 8\sqrt{3})$. The domain Ω with the initial mesh is shown in Figure 3a, and the "spurious solution" u is shown in Figure 3b. Here, $\omega = \angle Q_1QQ_2 = \frac{2\pi}{3} \in (\frac{\pi}{2}, \pi)$.

The direct mixed finite element solution u_{10}^U and the difference $|u-u_{10}^U|$ are shown in Figure 3c,d, respectively. The error $\|u - u_j^U\|_{H^1(\Omega)}$ is shown in Table 4. These results indicate that the direct mixed finite element solution converges to the "spurious solution" $u \notin H^3(\Omega)$. On the other hand, since $\omega = \frac{2\pi}{3} \in (\frac{\pi}{2}, \pi)$, so it follows $N = 1$ in Algorithm 3.1 by checking Table 1. The solution u_{10}^A from Algorithm 3.1and the difference $|u-u_{10}^A|$ are shown in Figure 3e,f, respectively. The error $\|u-u_j^A\|_{H^1(\Omega)}$ is shown in Table 4. These results imply that the solution of Algorithm 3.1 does not converge to the "spurious solution", since the solution of Algorithm 3.1 converges to the solution in $H^3(\Omega)$ as stated in Theorem 2.17.

Test case 2. Here, we consider the domain Ω to be the polygon with vertices $Q(0,0)$, $Q_1(\frac{16\sqrt{3}}{3}, 0)$, $Q_2(\frac{16-8\sqrt{2}}{1+\sqrt{3}}, \frac{16-8\sqrt{2}}{1+\sqrt{3}} + 8\sqrt{2})$, $Q_3(-8\frac{\sqrt{2}+\frac{2\sqrt{3}}{2}}{1+\frac{1}{\sqrt{3}}}, 8\sqrt{2} - 8\frac{\sqrt{2}+\frac{2\sqrt{3}}{2}}{1+\frac{1}{\sqrt{3}}})$ and $Q_4(-\frac{8}{3}, -4\sqrt{3})$. Then we have $\omega = \angle Q_1QQ_4 \approx 1.383\pi \in (\pi, \frac{3\pi}{2})$. The domain Ω with the initial mesh is shown in Figure 4a, and the "spurious solution" u is shown in Figure 4b.

The direct mixed finite element solution u_{10}^U and the difference $|u-u_{10}^U|$ are shown in Figure 4c,d, respectively. The error

TABLE 4 | The H^1 error of the numerical solutions on quasi-uniform meshes.

	$j=7$	$j=8$	$j=9$	$j=10$
$\|u-u_j^U\|_{H^1(\Omega)}$	2.74964e-01	1.35594e-01	6.77391e-02	3.38605e-02
$\|u-u_j^A\|_{H^1(\Omega)}$	6.07564	6.02331	6.00958	6.00306

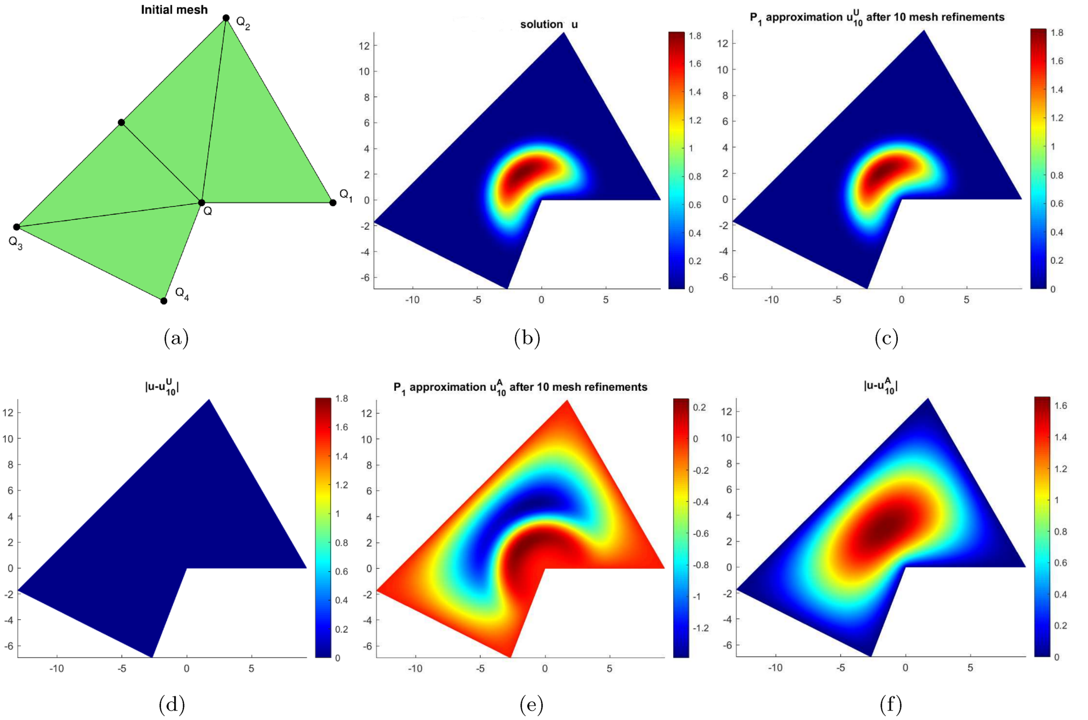


FIGURE 4 | Example 4.1 Test case 2: (a) the domain and the initial mesh; (b) the "spurious solution" u; (c) the direct mixed finite element solution u_{10}^U; (d) the difference $|u-u_{10}^U|$; (e) the solution u_{10}^A from Algorithm 3.1; (f) the difference $|u-u_{10}^A|$.

TABLE 5 | The H^1 error of the numerical solutions on quasi-uniform meshes.

	$j=7$	$j=8$	$j=9$	$j=10$
$\|u-u_j^U\|_{H^1(\Omega)}$	1.43517e-01	7.44186e-02	3.94988e-02	2.13310e-02
$\|u-u_j^A\|_{H^1(\Omega)}$	4.08611	4.08457	4.08383	4.08329

$\|u-u_j^U\|_{H^1(\Omega)}$ is shown in Table 5. These results imply that the direct mixed finite element solution converges to the "spurious solution" $u \notin H^3(\Omega)$. On the other hand, since $\omega \in (\pi, \frac{3\pi}{2})$, we have $N=2$ in Algorithm 3.1. The solution u_{10}^A of Algorithm 3.1 and the difference $|u-u_{10}^A|$ are shown in Figure 4e,f, respectively. The error $\|u-u_j^A\|_{H^1(\Omega)}$ is shown in Table 5. These results imply that the solution of Algorithm 3.1 does not converge to the "spurious solution."

Test case 3. Consider the polygonal domain Ω with vertices $Q(0,0)$, $Q_1(\frac{16\sqrt{3}}{3},0)$, $Q_2(\frac{16-8\sqrt{2}}{1+\sqrt{3}}, \frac{16-8\sqrt{2}}{1+\sqrt{3}}+8\sqrt{2})$, $Q_3(-8\frac{\sqrt{2}+\frac{2\sqrt{3}}{2}}{1+\frac{1}{\sqrt{3}}}, 8\sqrt{2}-8\frac{\sqrt{2}+\frac{2\sqrt{3}}{2}}{1+\frac{1}{\sqrt{3}}})$ and $Q_4(4,-8\sqrt{3})$. Then we have $\omega = \angle Q_1QQ_4 \approx 1.589\pi \in (\frac{3\pi}{2}, 2\pi)$. The domain Ω with the initial mesh is shown in Figure 5a, and the "spurious solution" u is shown in Figure 5b.

The direct mixed finite element solution u_{10}^U and the difference $|u-u_{10}^U|$ are shown in Figure 5c,d, respectively. The error $\|u-u_j^U\|_{H^1(\Omega)}$ is shown in Table 6. These results continue to indicate that the direct mixed finite element solution converges to the "spurious solution" $u \notin H^3(\Omega)$. On the other hand, since $\omega \in (\frac{3\pi}{2}, 2\pi)$, it follows $N=3$ in Algorithm 3.1. The solution u_{10}^A of Algorithm 3.1 and the difference $|u-u_{10}^A|$ are shown in Figure 5e,f, respectively. The error $\|u-u_j^A\|_{H^1(\Omega)}$ is shown in Table 6. These results confirm that the solution of Algorithm 3.1 does not converge to the "spurious solution."

Example 4.2. We solve the triharmonic problem in Example 4.1 again using the direct mixed finite element method and Algorithm 3.1 on quasi-uniform meshes. Here, we take the solution u_{ex} of the following Poisson problem as the exact solution,

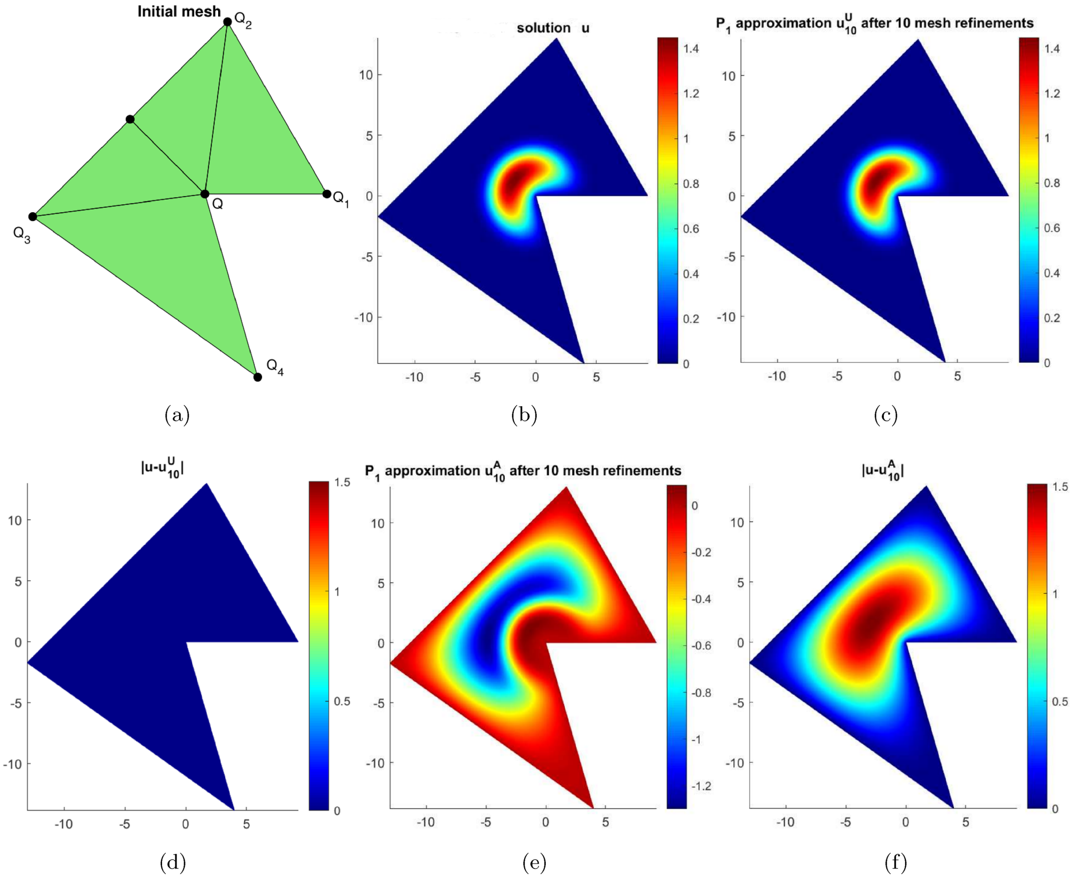


FIGURE 5 | Example 4.1 Test Case 3: (a) the domain and the initial mesh; (b) the "spurious solution" u; (c) the direct mixed finite element solution u_{10}^U; (d) the difference $|u - u_{10}^U|$; (e) the solution u_{10}^A from Algorithm 3.1; (f) the difference $|u - u_{10}^A|$.

TABLE 6 | The H^1 error of the numerical solutions on quasi-uniform meshes.

	$j = 7$	$j = 8$	$j = 9$	$j = 10$
$\|u - u_j^U\|_{H^1(\Omega)}$	1.474223e-01	8.67096e-02	5.25520e-02	3.25455e-02
$\|u - u_j^A\|_{H^1(\Omega)}$	3.863711	3.85981	3.85832	3.85767

$$-\Delta u_{ex} = f_0 - \sum_{i=1}^{N} c_i \sigma_i \text{ in } \Omega, \qquad u_{ex} = 0 \text{ on } \partial\Omega \tag{4.5}$$

where

$$f_0 = -\Delta\left(\tilde{\eta}(r;\tau,R) r^{\frac{N\pi}{\omega}} \sin\left(\frac{N\pi}{\omega}\theta\right)\right) \in H_0^1(\Omega),$$

with $\tilde{\eta}(r;\tau,R)$ given in (4.4), σ_i given in (2.18), and c_i is the solution of the linear system (2.39). Note that the function $f_0 = -\Delta u$ for u in (4.3). By Lemma 2.12, we have $u_{ex} \in H^3(\Omega)$ and it satisfies

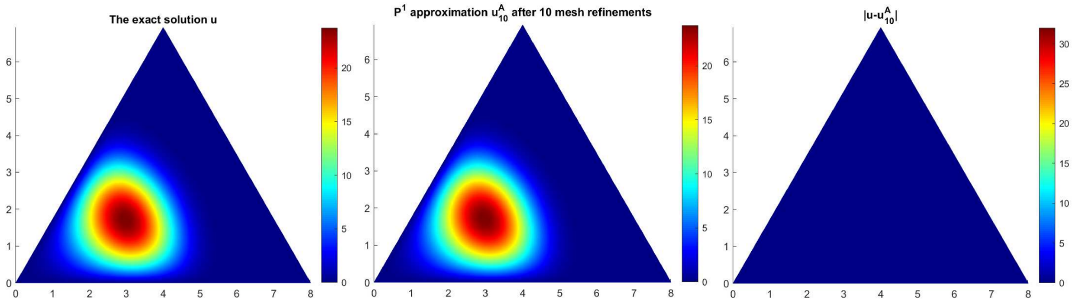
$$-\Delta^3 u_{ex} = -\Delta^2(\Delta u_{ex}) = \Delta^2 f_0 - \sum_{i=1}^{N} c_i \Delta^2 \sigma_i = \Delta^2 f_0 = -\Delta^2(\Delta u)$$
$$= -\Delta(\Delta(\Delta u)) = f,$$

where we have used the result in Lemma 2.8. Here, the source term f is the same as that in Example 4.1. The purpose of this example is to test the convergence of the direct mixed finite element method and Algorithm 3.1 to the exact solution u_{ex} in (4.5).

From Test case 2 to Test case 4, we will use the finite element method solution u_{exn} (instead of using the complicated notation $u_{ex,n+1}$) of (4.5) on mesh $\mathcal{T}_{n+1}$ as an approximation of u_{ex}.

Test case 1. Take Ω as the triangle $\triangle QQ_1Q_2$ with $Q(0,0)$, $Q_1(8,0)$ and $Q_2(4, 4\sqrt{3})$. In this case, the exact solution $u_{ex} = u$ for a given u in (4.3), and its contour is given in Figure 6a. Here, $\omega = \angle Q_1QQ_2 = \frac{\pi}{3} \in (0, \frac{\pi}{2})$. Thus, Algorithm 3.1 coincides with the direct mixed finite element method. The solution $u_{10}^A (= u_{10}^U)$ from Algorithm 3.1 and the difference $|u - u_{10}^A|$ are shown in Figure 6b,c, respectively. The error $\|u - u_j^A\|_{H^1(\Omega)}$ and convergence rate $\mathcal{R}$ are shown in Table 7. These results show that the solution of Algorithm 3.1 converges to the exact solution in the optimal convergence rate $\mathcal{R} = 1$, which coincides with the result in Theorem 3.13 or Table 3.

Test case 2. We consider the same domain and initial mesh (see Figure 3a) as Test case 1 in Example 4.1. Note that $\omega = \angle Q_1QQ_2 = \frac{2\pi}{3} \in (\frac{\pi}{2}, \pi)$. The finite element solution u_{exn} of the exact solution u_{ex} is shown in Figure 7a. The direct mixed

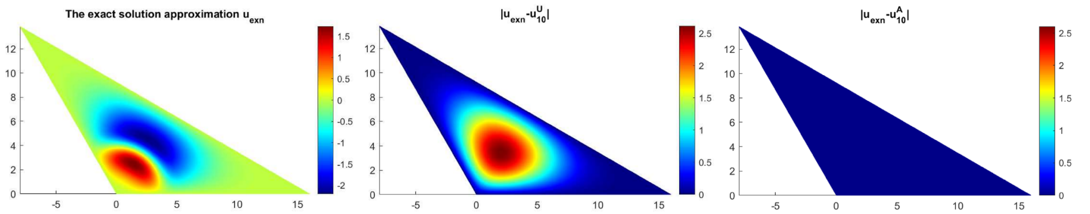


(a) (b) (c)

FIGURE 6 | Example 4.2 Test case 1: (a) the exact solution u; (b) the solution u_{10}^A from Algorithm 3.1; (c) the difference $|u - u_{10}^A|$.

TABLE 7 | The H^1 error and convergence rate $\mathcal{R}$ for Example 4.2 Test case 1.

	$j = 5$	$j = 6$	$j = 7$	$j = 8$
$\|u - u_j^A\|_{H^1(\Omega)}$	1.09202	5.45465e-01	2.72663e-01	1.36323e-01
$\mathcal{R}$	—	1.00	1.00	1.00



(a) (b) (c)

FIGURE 7 | Example 4.2 Test case 2: (a) the exact solution approximation u; (b) the difference $|u_{exn} - u_{10}^U|$; (c) the difference $|u_{exn} - u_{10}^A|$.

TABLE 8 | The H^1 error of the numerical solutions on quasi-uniform meshes.

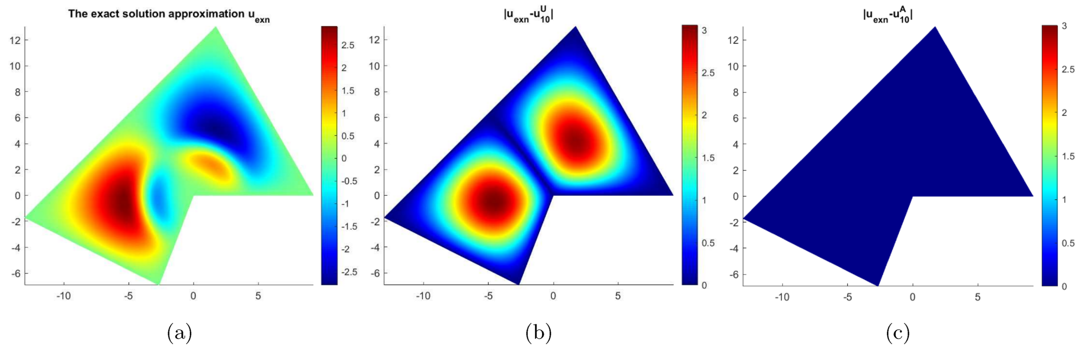
	$j = 6$	$j = 7$	$j = 8$	$j = 9$
$\|u_{exn} - u_j^U\|_{H^1(\Omega)}$	5.98206	6.01120	6.00363	5.99948
$\|u_{exn} - u_j^A\|_{H^1(\Omega)}$	5.67208e-02	1.47272e-02	6.62074e-03	3.43917e-03

finite element solution u_{10}^U and the difference $|u_{exn} - u_{10}^U|$ are shown in Figures 3c and 7b, respectively. The error $\|u_{exn} - u_j^U\|_{H^1(\Omega)}$ is shown in Table 8. These results indicate that the direct mixed finite element solution does not converge to the exact solution. Note that $N = 1$ in Algorithm 3.1, the solution u_{10}^A from Algorithm 3.1 and the difference $|u_{exn} - u_{10}^A|$ are shown in Figures 3e and 7c, respectively. The error $\|u_{exn} - u_j^A\|_{H^1(\Omega)}$ is shown in Table 8. These results imply that the solution of Algorithm 3.1 converges to the exact solution.

Test case 3. We consider the same domain and initial mesh (see Figure 4a) as Test case 2 in Example 4.1. Recall that $\omega = \angle Q_1 Q Q_4 \approx 1.383\pi \in (\pi, \frac{3\pi}{2})$. The exact solution u_{exn} is shown in Figure 8a. The direct mixed finite element solution u_{10}^U and the difference $|u_{exn} - u_{10}^U|$ are shown in Figures 4c and 8b, respectively. The error $\|u_{exn} - u_j^U\|_{H^1(\Omega)}$ is shown in Table 9. These results indicate that the direct mixed finite element solution does not converge to the exact solution. Note that $N = 2$ in Algorithm 3.1 in this case. The solution u_{10}^A of Algorithm 3.1 and the difference $|u_{exn} - u_{10}^A|$ are shown in Figures 4e and 8c, respectively. The error $\|u_{exn} - u_j^A\|_{H^1(\Omega)}$ is shown in Table 9. These results also imply that the solution of Algorithm 3.1 converges to the exact solution.

Test case 4. We consider the same domain and initial mesh (see Figure 5a) as Test case 3 in Example 4.1. Recall that $\omega = \angle Q_1 Q Q_4 \approx 1.589\pi \in (\frac{3\pi}{2}, 2\pi)$. The approximation u_{exn} of the exact solution is shown in Figure 9a. The direct mixed finite element solution u_{10}^U and the difference $|u_{exn} - u_{10}^U|$ are shown in Figures 5c and 9b, respectively. The error $\|u_{exn} - u_j^U\|_{H^1(\Omega)}$ is shown in Table 6. These results continue to indicate that the direct mixed finite element solution does not converge to the exact solution. Note that $N = 3$ in Algorithm 3.1, the solution u_{10}^A of Algorithm 3.1 and the difference $|u_{exn} - u_{10}^A|$ are shown in Figures 5e and 9c, respectively. The error $\|u_{exn} - u_j^A\|_{H^1(\Omega)}$ is shown in Table 10. These results confirm that the solution of Algorithm 3.1 converges to the exact solution.

Example 4.3. In this example, we investigate the convergence of Algorithm 3.1 by considering equation (1.1) with $f = \sin\left(\frac{N\pi}{\omega}\theta\right)$ on different domains with angle ω categorized in Theorem 3.13 or Table 3, where N is shown in Table 1. For $\omega < \frac{\pi}{2}$, the numerical test on convergence rate can be found in

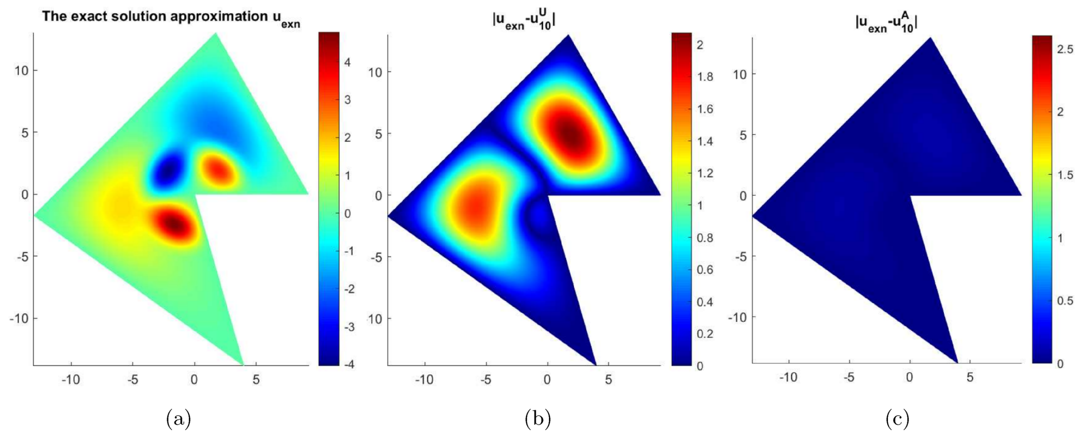


(a) (b) (c)

FIGURE 8 | Example 4.2 Test case 3: (a) the exact solution u; (b) the difference $|u_{exn} - u_{10}^{U}|$; (c) the difference $|u_{exn} - u_{10}^{A}|$.

TABLE 9 | The H^1 error of the numerical solutions on quasi-uniform meshes.

	$j = 6$	$j = 7$	$j = 8$	$j = 9$
$\|u_{exn} - u_j^U\|_{H^1(\Omega)}$	9.67666	9.64665	9.63404	9.63164
$\|u_{exn} - u_j^A\|_{H^1(\Omega)}$	5.27303e-02	2.09405e-02	1.01081e-02	4.20655e-03



(a) (b) (c)

FIGURE 9 | Example 4.2 Test case 4: (a) the exact solution u; (b) the difference $|u_{exn} - u_{10}^{U}|$; (c) the difference $|u_{exn} - u_{10}^{A}|$.

TABLE 10 | The H^1 error of the numerical solutions on quasi-uniform meshes.

	$j = 6$	$j = 7$	$j = 8$	$j = 9$
$\|u_{exn} - u_j^U\|_{H^1(\Omega)}$	7.47470	6.98223	6.60342	6.31031
$\|u_{exn} - u_j^A\|_{H^1(\Omega)}$	6.79611e-01	4.98616e-01	3.78626e-01	2.93364e-01

Example 4.2Test case 1. In the rest of this example, we focus on $\omega > \frac{\pi}{2}$.

Test case 1. Take Ω as the triangle $\triangle QQ_1Q_2$ with $Q(0,0)$, $Q_1(16,0)$ and $Q_2(16x_0, 16\sqrt{1-x_0^2})$ for some $|x_0| < 1$. The convergence rates for different $\omega = \angle Q_1QQ_2 \in (\frac{\pi}{2}, \pi)$ determined by choosing different x_0 are shown in Table 11. Here, $R = \frac{24}{5}$, $\tau = \frac{1}{8}$ are used when $x_0 = -0.8$, and default values are used for other cases. The results show that the convergence rate is not optimal when $\omega < \frac{2\pi}{3}$, and it is optimal when $\omega \in [\frac{2\pi}{3}, \pi)$. These results are consistent with the expected convergence rate $\mathcal{R}$ in Theorem 3.13 or Table 3 for $\omega \in (\frac{\pi}{2}, \pi)$.

Test case 2. We consider the polygon Ω with vertices $Q(0,0)$, $Q_1(16,0)$, $Q_2(-8, 8\sqrt{3})$, and $Q_3(-8, -8y_0\sqrt{3})$ for some $y_0 \in (0,1]$, which gives $\omega = \angle Q_1QQ_3 \in (\pi, \frac{4\pi}{3}]$. We then consider the domain Ω (see Figure 5a) presented in Example 4.1 Test case 2, and the corresponding angle $\omega \in (\frac{4\pi}{3}, \frac{3\pi}{2})$. The convergence rates for different $\omega \in (\pi, \frac{3\pi}{2})$ are shown in Table 12. The results show that the convergence rate is not optimal when $\omega < \frac{4\pi}{3}$, and it is optimal when $\omega \in [\frac{4\pi}{3}, \frac{3\pi}{2})$. These results are consistent with the expected convergence rate in Theorem 3.13 or Table 3 for $\omega \in (\pi, \frac{3\pi}{2})$.

Test case 3. We consider the polygon Ω with vertices $Q(0,0)$, $Q_1(\frac{16\sqrt{3}}{3}, 0)$, $Q_2(\frac{16-8\sqrt{2}}{1+\sqrt{3}}, \frac{16-8\sqrt{2}}{1+\sqrt{3}} + 8\sqrt{2})$, $Q_3(-8\frac{\sqrt{2}+\frac{2\sqrt{3}}{2}}{1+\frac{1}{\sqrt{3}}}, 8\sqrt{2} - 8\frac{\sqrt{2}+\frac{2\sqrt{3}}{2}}{1+\frac{1}{\sqrt{3}}})$, and $Q_4(x_1, -8\sqrt{3})$ for some $x_1 \in (0, 8\sqrt{3}]$, which generates $\omega = \angle Q_1QQ_4 \in (\frac{3\pi}{2}, \frac{7\pi}{4}]$. The convergence rates for different $\omega \in (\frac{3\pi}{2}, 2\pi)$ are shown in Table 13. These results are consistent with the expected convergence rate in Theorem 3.13 or Table 3 for $\omega \in (\frac{3\pi}{2}, 2\pi)$.

TABLE 11 | The H^1 error for $\omega \in (\frac{\pi}{2}, \pi)$ on quasi-uniform meshes.

Parameter x_0	ω	Expected rate	$j = 7$	$j = 8$	$j = 9$	$j = 10$
−0.2	$\approx 0.56409\pi$	0.46	0.75	0.67	0.59	0.54
−0.4	$\approx 0.63099\pi$	0.83	0.96	0.95	0.94	0.93
−0.5	$\frac{2\pi}{3}$	1.00	1.03	1.01	1.00	1.00
−0.6	$\approx 0.70483\pi$	1.00	1.01	1.01	1.01	1.00
−0.8	$\approx 0.79517\pi$	1.00	1.02	1.01	1.01	1.00

TABLE 12 | The H^1 error for $\omega \in (\pi, \frac{3\pi}{2})$ on quasi-uniform meshes.

Parameter y_0 or domain	ω	Expected rate	$j = 6$	$j = 7$	$j = 8$	$j = 9$
0.2	$\approx 1.10615\pi$	0.38	0.82	0.72	0.62	0.53
0.6	$\approx 1.25612\pi$	0.82	0.96	0.95	0.94	0.93
0.8	$\approx 1.30101\pi$	0.93	0.98	0.98	0.98	0.98
1.0	$\frac{4\pi}{3}$	1.00	1.00	1.00	1.00	1.00
Ω in Figure 5a	$\approx 1.38305\pi$	1.00	1.02	1.02	1.01	1.01

TABLE 13 | The H^1 error for $\omega \in (\frac{3\pi}{2}, 2\pi)$ on quasi-uniform meshes.

x_1 or domain	ω	Expected rate	$j = 6$	$j = 7$	$j = 8$	$j = 9$
$x_1 = 4$	$\approx 1.58946\pi$	0.23	0.87	0.76	0.63	0.50
$x_1 = 8$	$\frac{5\pi}{3}$	0.40	0.83	0.75	0.65	0.60
$x_1 = 8\sqrt{3}$	$\frac{7\pi}{4}$	0.57	0.87	0.82	0.77	0.71

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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